1.3 Vector Equations

McDonald Fall 2018, MATH 2210Q 1.3 Slides

Homework: Read the section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 6, 9, 11, 15, 21, 23, 25
- Extra Practice: 3, 9, 12, 14, 22

Definition 1.3.1 (Vectors in $\mathbb{R}^2$)**.** A matrix with only one column is called a **column vector**, or just a **vector**. Examples of vectors with two entries are

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} \sqrt{2} \\ \pi \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

where w_1, w_2 are real numbers. The set of all vectors with two entries is called $\mathbb{R}^2$. Two vectors are **equal** if and only if their corresponding entries are equal.

Definition 1.3.2. Given two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^2$, their **sum** is the vector $\mathbf{u}+\mathbf{v}$ obtained by adding the corresponding entries of $\mathbf{u}$ and $\mathbf{v}$. For example,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+2 \\ 2+3 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

Given a vector $\mathbf{v}$ and a real number c, the **scalar multiple** of $\mathbf{u}$ is the vector $c\mathbf{u}$ obtained by multiplying each entry of $\mathbf{u}$ by c. For example if

$$c = 2 \text{ and } \mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \text{ then } c\mathbf{u} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Example 1.3.3. Given vectors $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, find $(-2)\mathbf{u}$, $(-2)\mathbf{v}$, and $\mathbf{u} + (-3)\mathbf{v}$.

Observation 1.3.4 (Vectors in $\mathbb{R}^2$)**.** We can identify the column vector $\left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$ with *the point* (a, b) in the plain, so we can consider $\mathbb{R}^2$ as the set of all points in the plain. We usually visualize a vector by including an arrow from the origin.

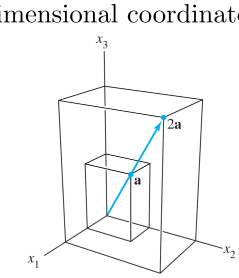
Example 1.3.5. Let $\mathbf{u} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -6 \\ 1 \end{bmatrix}$. Graph $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ on the plane.

Proposition 1.3.6 (Parallelogram Rule)**.** *If* $\mathbf{u}$ *and* $\mathbf{v}$ *in* $\mathbb{R}^2$ *are represented in the plain, then* $\mathbf{u} + \mathbf{v}$ *corresponds to the last vertex of the parallelogram with vertices are* $\mathbf{u}$, $\mathbf{v}$ *and* $\mathbf{0}$.

Example 1.3.7. Let $\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Graph $\mathbf{u}$, $(-2)\mathbf{u}$, and $3\mathbf{u}$. What's special about $c\mathbf{u}$ for any c?

Observation 1.3.8 (Vectors in $\mathbb{R}^3$)**.** Vectors in $\mathbb{R}^3$ are 3×1 matrices. Like above, we can represent them geometrically in three-dimensional coordinate space. For example,

$$\mathbf{a} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$



Definition 1.3.9 (Vectors in $\mathbb{R}^n$)**.** If n is a positive integer, $\mathbb{R}^n$ denotes the collection of ordered n-tuples of n real numbers, usually written as $n \times 1$ column matrices, such as

$$\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix},$$

we we again, sometimes denote $(a_1, a_2, \ldots, a_n)$. The **zero vector**, denoted $\mathbf{0}$ is the vector whose entries are all zero. We also denote $(-1)\mathbf{u} = -\mathbf{u}$.

Proposition 1.3.10 (Algebraic Properties of $\mathbb{R}^n$)**.** *For* $\mathbf{u}$*,* $\mathbf{v}$*,* $\mathbf{w}$ *in* $\mathbb{R}^n$*, and scalars* c*,* d*:*

(i) $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$	(v) $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$
(ii) $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$	(vi) $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$
(iii) $\mathbf{u}+\mathbf{0}=\mathbf{0}+\mathbf{u}=\mathbf{u}$	(vii) $c(d\mathbf{u})=(cd)\mathbf{u}$
(iv) $\mathbf{u}+(-\mathbf{u})=-\mathbf{u}+\mathbf{u}=\mathbf{0}$	(viii) $1\mathbf{u}=\mathbf{u}$

Remark 1.3.11. Sometimes, for ease of notation, we denote $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$ as $(a_1, a_2, \ldots, a_n)$.

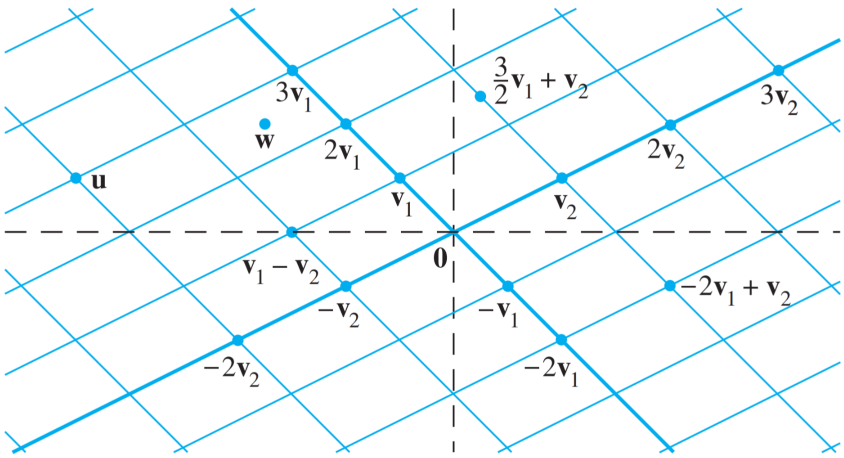
Example 1.3.12. Prove properties (i) and (v) of the Algebraic Properties above.

Definition 1.3.13 (Linear Combinations)**.** Given vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ in $\mathbb{R}^n$, and scalars $c_1, c_2, \ldots, c_m$. The vector

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m$$

is called a **linear combination** of the $\mathbf{v}_1, \ldots \mathbf{v}_m$ with **weights** $c_1, \ldots, c_m$.

Example 1.3.14. The figure below shows linear combinations of $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ where with integer weights. Estimate the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that produce $\mathbf{u}$ and $\mathbf{w}$.



Example 1.3.15. Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ -5 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ 4 \\ -3 \end{bmatrix}$. Is $\mathbf{b}$ a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$?

Remark 1.3.16. In the previous example, the vectors $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{b}$ became the columns of the augmented matrix that we reduced:

$$\begin{bmatrix} 1 & 2 & 7 \\ -2 & 5 & 4 \\ -5 & 6 & -3 \end{bmatrix}$$

For brevity, we will write this matrix, using vectors, as $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix}$. This suggests the following.

Procedure 1.3.17. A vector equation $x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$, has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}$$

In particular, $\mathbf{b}$ can be represented as a linear combination of $\mathbf{a}_1, \ldots, \mathbf{a}_n$ if and only if there is a solution to the linear system corresponding to this matrix.

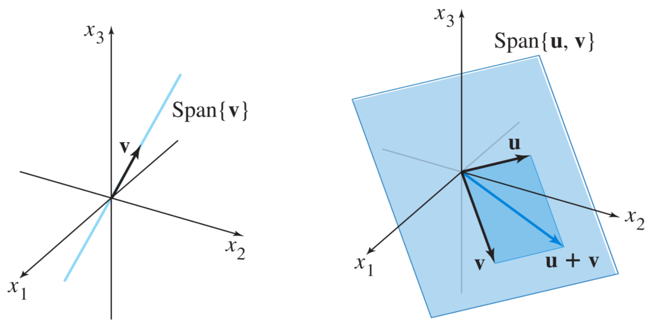
Definition 1.3.18. If $\mathbf{v}_1, \ldots, \mathbf{v}_m$ are in $\mathbb{R}^n$, then the set of all linear combinations of is denoted by $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ and is called the **subset of $\mathbb{R}^n$ spanned by $\mathbf{v}_1, \ldots, \mathbf{v}_m$**. In other words, $\mathrm{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is the collection of all vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_m\mathbf{v}_m, \text{ with } c_1, \ldots, c_m \text{ scalars.}$$

Example 1.3.19. Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Prove that $\mathbf{v}_1$ and $\mathbf{v}_2$ span all of $\mathbb{R}^2$.

Remark 1.3.20. Actually, for *any* $\mathbf{u}$ and $\mathbf{v}$ (which are not multiples) in $\mathbb{R}^3$, $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is a plane!

Observation 1.3.21 (Geometric Descriptions of $\text{Span}\{\mathbf{u}\}$ and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$)**.** Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^3$, with $\mathbf{u}$ not a multiple of $\mathbf{v}$. Then $\text{Span}\{\mathbf{v}\}$ is the set of points on the line in $\mathbb{R}^3$ through $\mathbf{0}$ and $\mathbf{v}$, and $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ is the plane in $\mathbb{R}^3$ containing $\mathbf{0}$, $\mathbf{u}$ and $\mathbf{v}$, that is, it contains the line in $\mathbb{R}^3$ through $\mathbf{u}$ and the line through $\mathbf{0}$ and $\mathbf{v}$ and $\mathbf{0}$.



Example 1.3.22. If $\mathbf{a}_1 = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 5 \\ -13 \\ -3 \end{bmatrix}$. Is $(-3, 8, 1)$ in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$?