NOTE: These slides contain *both* Section 1.4 and 1.5.

1.4 The Matrix Equation $A\mathbf{x} = \mathbf{b}$

McDonald Fall 2018, MATH 2210Q 1.4 &1.5 Slides

1.4 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 1, 3, 13, 17, 19, 22, 23, 25
- Extra Practice: 4, 7, 9, 11, 31

The definition below lets us rephrase some of the concepts from Section 1.3 by viewing linear combinations of vectors as the product of a matrix and a vector

Definition 1.4.1. If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$, and $\mathbf{x}$ is in $\mathbb{R}^n$, then the **product of** A **and** $\mathbf{x}$**,** denoted by $A\mathbf{x}$, is the linear combination of the columns of A using the corresponding entries in $\mathbf{x}$ as weights:

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n.$$

Remark 1.4.2. $A\mathbf{x}$ is only defined the number of columns of A equals the number of entries in $\mathbf{x}$.

Example 1.4.3. Find the following products:

(a) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -5 & 3 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 9 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Example 1.4.4. Compute $A\mathbf{x}$, where $A = \begin{bmatrix} 2 & 3 & 4 \\ -1 & 5 & -3 \\ 6 & -2 & 8 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

> **Procedure 1.4.5** (Row Vector Rule)**.** If $A\mathbf{x}$ is defined, then the ith entry in $A\mathbf{x}$ is the sum of the products of corresponding entries from row i of A and from the vector $\mathbf{x}$.

Example 1.4.6. Compute

(a) $\begin{bmatrix} 2 & -3 \\ 8 & 0 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

Example 1.4.7. Write the system below as $A\mathbf{x} = \mathbf{b}$ for some A and $\mathbf{b}$.

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ -5x_2 + 3x_3 &= 1 \end{aligned}$$

Definition 1.4.8. The equation $A\mathbf{x} = \mathbf{b}$ is called a **matrix equation.**

Theorem 1.4.9. *If A is an $m \times n$ matrix, with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$, and $\mathbf{b}$ is in $\mathbb{R}^m$, then*

$$A\mathbf{x} = \mathbf{b} \ (\textit{with } \mathbf{x} \textit{ in } \mathbb{R}^n)$$

has the same solutions as the vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

which has the same solutions as the system of linear equations with augmented matrix

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix}.$$

Corollary 1.4.10. *The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is a linear combination of the columns of A.*

Example 1.4.11. Let $A = \begin{bmatrix} 1 & 3 & 4 \\ -4 & 2 & -6 \\ -3 & -2 & -7 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ is $A\mathbf{x} = \mathbf{b}$ consistent for all b_1, b_2, b_3?

Theorem 1.4.12. *Let A be an $m \times n$ matrix. Then the following statements are either all true, or all false.*

(a) *For each $\mathbf{b}$ in $\mathbb{R}^m$, the equation $A\mathbf{x} = \mathbf{b}$ has a solution.*

(b) *Each $\mathbf{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A.*

(c) *The columns of A span $\mathbb{R}^m$.*

(d) *A has a pivot position in every row.*

Example 1.4.13. If $A = \begin{bmatrix} 2 & 0 & -2 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 11 \\ 3 \end{bmatrix}$, for what $\mathbf{x}$ is $A\mathbf{x} = \mathbf{b}$ consistent?

We end this section with some important properties of $A\mathbf{x}$, which we will use throughout the course.

Theorem 1.4.14. *If A is an $m \times n$ matrix, $\mathbf{u}$ and $\mathbf{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar:*

(a) $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v}$*;*

(b) $A(c\mathbf{u}) = c(A\mathbf{u})$*.*

1.5 Solutions Sets of Linear Systems

1.5 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 5, 11, 15, 19, 23, 30, 32
- Extra Practice: 2, 6, 18, 22, 27

In this section, we will use vector notation to give explicit and geometric descriptions of solution sets of linear systems. We begin by defining a special type of system.

Definition 1.5.1. A system of linear equations is said to be **homogeneous** if it can be written in the form $A\mathbf{x} = \mathbf{0}$, where A is an $m \times n$ matrix, and $\mathbf{0}$ is the zero vector in $\mathbb{R}^m$.

Remark 1.5.2. The equation $A\mathbf{x} = \mathbf{0}$ *always* has at least one solution, namely $\mathbf{x} = \mathbf{0}$, called the **trivial solution**. We will be interested in finding **non-trivial solutions**, where $\mathbf{x} \neq \mathbf{0}$.

Example 1.5.3. Determine if the following homogeneous system has a nontrivial solution, and describe the solution set.

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 0 \\ -3x_1 - 2x_2 + 4x_3 &= 0 \\ 6x_1 + x_2 - 8x_3 &= 0 \end{aligned}$$

Proposition 1.5.4. *The homogeneous equation $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution if and only if the equation has at least one free variable.*

Example 1.5.5. Describe all solutions to the homogeneous system

$$10x_1 - 3x_2 - 2x_3 = 0.$$

Definition 1.5.6. The answers in 1.5.3 and 1.5.5 are **parametric vector equations**. Sometimes, to emphasize that the parameters vary over all real numbers, we write

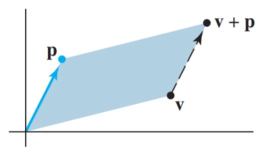
$$\mathbf{x} = s\mathbf{u} + t\mathbf{v} \text{ for } s, t \in \mathbb{R}.$$

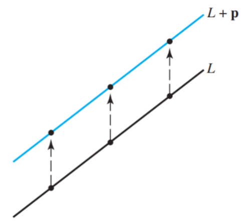
In both examples, we say that the solution is in **parametric vector form.**

Example 1.5.7. Describe all solutions of

$$\begin{aligned} 3x_1 + 5x_2 - 4x_3 &= 7 \\ -3x_1 - 2x_2 + 4x_3 &= -1 \\ 6x_1 + x_2 - 8x_3 &= -4 \end{aligned}$$

Definition 1.5.8. We can think of vector addition as *translation*. Given $\mathbf{p}$ and $\mathbf{v}$ in $\mathbb{R}^2$ or $\mathbb{R}^3$, the effect of adding $\mathbf{p}$ to $\mathbf{v}$ is to *move* v in a direction parallel to the line through $\mathbf{p}$ and $\mathbf{0}$. We say that $\mathbf{v}$ is **translated by p** to $\mathbf{v}+\mathbf{p}$. If each point on a line L is translated by a vector $\mathbf{p}$, the result is a line parallel to L.

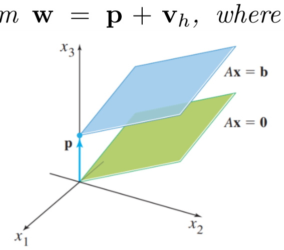




For $t \in \mathbb{R}$, we call $\mathbf{p}+t\mathbf{v}$ **the equation of the line parallel to v through p.**

Example 1.5.9. Use this observation to describe the relationships between the solutions to $A\mathbf{x}=\mathbf{0}$ and $A\mathbf{x}=\mathbf{b}$ using the A and $\mathbf{b}$ from Examples 1.5.3 and 1.5.7.

Theorem 1.5.10. *Suppose the equation* $A\mathbf{x}=\mathbf{b}$ *is consistent for some given* $\mathbf{b}$*, and let* $\mathbf{p}$ *be a solution. Then the solution set of* $A\mathbf{x}=\mathbf{b}$ *is the set of all vecotrs of the form* $\mathbf{w}=\mathbf{p}+\mathbf{v}_h$*, where* $\mathbf{v}_h$ *is any solution of the homogeneous equation* $A\mathbf{x}=\mathbf{0}$.



Procedure 1.5.11. To write a solution set in parametric vector form

1. Row reduce the augmented matrix to RREF
2. Express each basic variable in terms of any free variables
3. Write $\mathbf{x}$ as a vector whose entries depend on the free variables (if there are any)
4. Decompose $\mathbf{x}$ into a linear combination of vectors using free variables as parameters

Example 1.5.12. Describe and compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ if

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$