1.7 Linear Independence

McDonald Fall 2018, MATH 2210Q, 1.7Slides

1.7 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 1, 5, 7, 15, 16, 20, 21
- Extra Practice: 1-20

Definition 1.7.1. An indexed set of vectors $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in $\mathbb{R}^n$ is said to be **linearly independent** if the vector equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution. S is **linearly dependent** if for some $c_1, \ldots, c_p$ not all zero

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0}.$$

Example 1.7.2. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 3 \\ 3 \end{bmatrix}$.
Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? If not, find a linear dependence relation.

Example 1.7.3. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.
Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent? If not, find a linear dependence relation.

Remark 1.7.4. If $A = \begin{bmatrix} \mathbf{v}_m & \cdots & \mathbf{v}_m \end{bmatrix}$, then the homogeneous equation $A\mathbf{x} = \mathbf{0}$ can be written

$$x_1\mathbf{v}_1 + \cdots + x_n\mathbf{v}_m = \mathbf{0}.$$

Thus, linear independence is the same as having no non-trivial solutions to this matrix equation.

Definition 1.7.5. The columns of a matrix A are linearly independent if and only if $A\mathbf{x} = \mathbf{0}$ has no non-trivial solutions.

Example 1.7.6. Determine if the columns of $A = \begin{bmatrix} 0 & 1 & 4 \\ 1 & 2 & -1 \\ 5 & 8 & 0 \end{bmatrix}$ are linearly independent.

Example 1.7.7. Determine if the following sets of vectors are linearly independent.

(a) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

(b) $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$

Proposition 1.7.8 (Sets of two vectors)**.** *A set of two vectors* $\{\mathbf{v}_1, \mathbf{v}_2\}$ *is linearly independent if and only if neither of the vectors is a multiple of the other.*

Theorem 1.7.9 (Characterization of Linearly Dependent Sets)**.** *An indexed set* $\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ *of two or more vectors is linearly dependent if and only if at least one of the vectors is a linear combination of the others. In fact, if* S *is linearly dependent and* $\mathbf{v}_1 \neq \mathbf{0}$*, then some* $\mathbf{v}_j$ $(j > 1)$ *is a linear combination of the preceding vectors,* $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example 1.7.10. If $\mathbf{u}$ and $\mathbf{v}$ are linearly independent non-zero vectors in $\mathbb{R}^3$. Geometrically describe $\text{Span}\{\mathbf{u}, \mathbf{v}\}$. Prove $\mathbf{w}$ is in $\text{Span}\{\mathbf{u}, \mathbf{v}\}$ if and only if $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a linearly dependent set.

Theorem 1.7.11 (Too many vectors)**.** *If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ *in* $\mathbb{R}^n$ *is linearly dependent if* $p > n$.

Proof:

Theorem 1.7.12. *If a set* S *in* $\mathbb{R}^n$ *contains the zero vector, then* S *is linearly dependent.*

Proof:

Example 1.7.13. Determine by inspection (without matrices) if given sets are linearly dependent.

(a) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$

(d) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$