## 1.8 Intro to Linear Transformations

McDonald Fall 2018, MATH 2210Q, 1.8Slides

**1.8 Homework**: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in:*** 2, 8, 9, 21, 31
- Extra Practice: 1-18

**Example 1.8.1.** If $A = \begin{bmatrix} 4 & -3 & 1 & 3 \\ 2 & 0 & 5 & 1 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \\ 3 \end{bmatrix}$. Find $A\mathbf{0}$, $A\mathbf{u}$, and $A\mathbf{v}$.

We can think of $A$ as *acting* on $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ like a *function* from one set of vectors to another.

**Definition 1.8.2.** A **transformation** (also called a **function** or **mapping**) $T$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a rule that assigns to each vector $\mathbf{x}$ in $\mathbb{R}^n$ one (and only one) vector $T(\mathbf{x})$ in $\mathbb{R}^m$. The set $\mathbb{R}^n$ is called the **domain** of $T$ and $\mathbb{R}^m$ is called the **codomain** of $T$, denoted

$$T : \mathbb{R}^n \to \mathbb{R}^m.$$

For $\mathbf{x}$ in $\mathbb{R}^n$, the vector $T(\mathbf{x})$ in $\mathbb{R}^m$ is called the **image** of $\mathbf{x}$. The subset of $\mathbb{R}^m$ consisting of all possible images $T(\mathbf{x})$ is called the **range**.

**Remark 1.8.3.** In this section, we will focus on mappings associated to *matrix multiplication*. For simplicity, we sometimes denote this *matrix transformation* by $\mathbf{x} \mapsto A\mathbf{x}$.

**Example 1.8.4.** Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and define a transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

(a) Write $T(\mathbf{x})$ as a vector.

(b) Find $T(\mathbf{u})$, the image of $\mathbf{u}$ under the transformation $T$.

(c) Find an $\mathbf{x}$ in $\mathbb{R}^2$ such that $T(\mathbf{x}) = \mathbf{b}$. Is there more than one?

(d) Determine if $\mathbf{c}$ is in the range of $T$.

**Example 1.8.5.** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and describe $\mathbf{x} \mapsto A\mathbf{x}$.

**Remark 1.8.6.** The map in Example 1.8.5 is called a **projection map.**

**Example 1.8.7.** Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$, and $T(\mathbf{x}) = A\mathbf{x}$. Find the images of $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under $T$, and use this to describe $T$ geometrically.

**Remark 1.8.8.** The map in Example 1.8.7 is called a **shear transformation**.

**Definition 1.8.9.** A transformation $T$ is called **linear** if

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of $T$;

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars $c$ and $\mathbf{u}$ in the domain of $T$.

**Remark 1.8.10.** The properties of $A\mathbf{x}$ Section 1.4 show that when $T$ is a *matrix transformation*, $T$ is a linear transformation. Not all transformations are linear, however.

**Proposition 1.8.11.** *If $T$ is a linear transformation, then*

$$T(\mathbf{0}) = \mathbf{0},$$

*and*

$$T(c\mathbf{u} + d\mathbf{v}) = cT(\mathbf{u}) + dT(\mathbf{v})$$

*for all vectors $\mathbf{u}, \mathbf{v}$ in the domain of $T$, and all scalars $c, d$.*

**Remark 1.8.12.** If $T$ satisfies the second property above, then it is as linear transformation. Repeated application of this property gives

$$T(c_1\mathbf{u}_1 + \cdots + c_n\mathbf{v}_n) = c_1T(\mathbf{u}_1) + \cdots + c_nT(\mathbf{v}_n)$$

In physics and engineering, this is called a *superposition principle*.

**Example 1.8.13.** Given a scalar $r \geq 0$, define $T : \mathbb{R}^2 \to \mathbb{R}^2$ by $T(\mathbf{x}) = r\mathbf{x}$. Prove $T$ is a linear transformation, and describe it geometrically.

**Remark 1.8.14.** In Example 1.8.12, the linear transformation $T$ is called a **contraction** when $0 < r < 1$ and a **dilation** when $r > 1$.

**Example 1.8.15.** Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $T(\mathbf{x}) = A\mathbf{x}$. Find the images of $\mathbf{u} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{u} + \mathbf{v} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$ under $T$, and use this to describe $T$ geometrically.

### 1.8.1 Additional Thoughts and Problems