2.1 Matrix Operations

McDonald Fall 2018, MATH 2210Q, 2.1 Slides

2.1 Homework: Read section and do the reading quiz. Start with practice problems, then do

- ***Hand in***: 2, 5, 7, 10, 15.
- Recommended: 20, 22, 27, 28.

Definition 2.1.1. If A is an $m \times n$ matrix (m rows and n columns), then the entry in the ith row and jth column of A, typically denoted a_{ij}, is called the (i, j)**-entry** of A. We write $A = [a_{ij}]$ using this notation. Columns of A are vectors in $\mathbb{R}^m$, usually denoted $\mathbf{a}_1, \ldots \mathbf{a}_n$. We often write:

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \end{bmatrix}.$$

The **diagonal entries** of $A = [a_{ij}]$ are $a_{11}, a_{22}, a_{33}, \ldots$, and they form the **main diagonal** of A. A **diagonal matrix** is an $n \times n$ square matrix whose nondiagonal entries are all zero. A **zero matrix** is an $m \times n$ matrix whose entries are all zero.

Definition 2.1.2. Two matrices are **equal** if they have the same size and their corresponding entires are equal. If A and B are matrices of the same size, then the **sum** $A + B$ is the matrix whose entries are the sums of the corresponding entries in A and B.

Example 2.1.3. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$, and $C = \begin{bmatrix} 1 & 3 \\ 5 & -6 \end{bmatrix}$.
Find $A + B$, $B + A$, and $A + C$.

Definition 2.1.4. If r is a scalar and A is a matrix, then the **scalar multiple** rA is the matrix whose entries are r times the corresponding entries of A. Notationally, $-A$ stands for $(-1)A$, and $A - B = A + (-1)B$.

Example 2.1.5. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ -4 & 5 & -6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 7 & -8 & 9 \end{bmatrix}$. Find $2B$ and $A - 2B$.

Theorem 2.1.6. *Let A, B, and C be matrices of the same size, and r and s be scalars.*

a. $A + B = B + A$

b. $(A + B) + C = A + (B + C)$

c. $A + 0 = A$.

d. $r(A + B) = rA + rB$

e. $(r + s)A = rA + rB$

f. $r(sA) = (rs)A$.

Definition 2.1.7. If A is an $m \times n$ matrix, and B is an $n \times p$ matrix with columns $\mathbf{b}_1, \ldots, \mathbf{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\mathbf{b}_1, \ldots, A\mathbf{b}_p$. That is

$$AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \cdots & A\mathbf{b}_p \end{bmatrix}.$$

Remark 2.1.8. If the number of columns of A doesn't match the number of rows of B, then the product AB is *undefined.*

Example 2.1.9. Compute AB and BA, when $A = \begin{bmatrix} 2 & 1 \\ -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 5 & 1 \\ 2 & -8 & 3 \end{bmatrix}$.

Procedure 2.1.10 (Row-Column Rule for AB)**.** If the product AB is defined, then the (i, j)-entry of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i, j)-entry in AB, and A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \cdots + a_{in}b_{nj}$$

Example 2.1.11. With A and B from Example 2.1.9, compute AB using the row-column rule.

Theorem 2.1.12. *Let A be an $m \times n$ matrix, and let B and C have the right sizes so that the following sums and products are defined.*

a. $A(BC) = (AB)C$

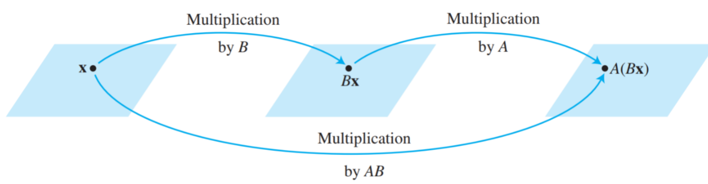
b. $A(B + C) = AB + AC$

c. $(B + C)A = BA + CA.$

d. $r(AB) = (rA)B = A(rB)$ *(for any scalar r)*

e. $I_m A = A = AI_n$

Remark 2.1.13. When a matrix B multiplies a vector $\mathbf{x}$, it transforms $\mathbf{x}$ into $B\mathbf{x}$. If this vector is multiplied by a second matrix A, the resulting vector is $A(B\mathbf{x})$. We can think about this as a *composition* of mappings. The matrix product is defined in a special way so that $A(B\mathbf{x}) = (AB)\mathbf{x}$.



Example 2.1.14. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation that first reflects points through the horizontal x_1-axis, and then reflects them through the line $x_2 = x_1$. Find the standard matrix of T.

Example 2.1.15. Let $A = \begin{bmatrix} 2 & -3 \\ -4 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \end{bmatrix}$, $C = \begin{bmatrix} 5 & -2 \\ 3 & 1 \end{bmatrix}$, and $D = \begin{bmatrix} 3 & 9 \\ 2 & 6 \end{bmatrix}$.

(a) Find AB and BA.

(b) Find AC.

(c) Find AD.

Watchout! 2.1.16. Here are some important warnings for matrix multiplication:

1. In general, $AB \neq BA$.
2. Cancellation laws *do not hold* for multiplication; $CA = CB$ (or $AC = BC$) does not mean $A = B$.
3. If $AB = 0$, this *does not mean* $A = 0$ or $B = 0$.

Definition 2.1.17. If A is an $n \times n$ square matrix and k is a positive integer, then we denote

$$A^k = AA \cdots A \ (k \text{ times})$$

We adopt the convention that $A^0 = I_n$.

Definition 2.1.18. If A is an $m \times n$ matrix, the **transpose** of A is the $n \times m$ matrix, denoted A^T, whose columns are formed from the corresponding rows of A.

Example 2.1.19. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $B = \begin{bmatrix} 8 & 4 \\ 5 & 5 \\ 6 & 2 \end{bmatrix}$, and $C = \begin{bmatrix} 5 & -2 & 1 & 3 \\ 3 & 1 & 2 & -6 \end{bmatrix}$.

Find A^T, B^T, and C^T.

Theorem 2.1.20. *Let A and B be matrices who are the right size for the following operations.*

a. $(A^T)^T = A$

b. $(A + B)^T = A^T + B^T$

c. $(rA)^T = rA^T$ *(for any scalar r)*

d. $(AB)^T = B^T A^T$