NOTE: These slides contain *both* Section 2.2 and 2.3.

2.2 The Inverse of a Matrix

McDonald Fall 2018, MATH 2210Q, 2.2&2.3 Slides

2.2 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 3, 6, 7, 9, 13, 29
- Recommended: 7, 11, 15, 23, 24, 32, 37

Definition 2.2.1. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I, \text{ where } I = I_n \text{ is the identity matrix.}$$

In this case, C is called the **inverse** of A. A matrix that is *not* invertible is called a **singular matrix**, and an invertible matrix is called a **non-singular matrix**.

Remark 2.2.2. Suppose B and C were both inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

It turns out, that if A has an inverse, it's unique. We call this unique inverse A^{-1}.

Example 2.2.3. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Show that $C = A^{-1}$

Theorem 2.2.4. *Invertible matrices have the following three properties.*

1. *If A is an invertible matrix, then A^{-1} is invertible, and $(A^{-1})^{-1} = A$.*
2. *If A and B are $n \times n$ invertible matrices, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$.*
3. *If A is an invertible matrix, then so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Theorem 2.2.5. *Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. *If* $ad - bc \neq 0$, *then* A *is invertible and*

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, *then* A *is not invertible.*

Remark 2.2.6. The quantity $ad - bc$ is called the **determinant** of A, and we write

$$\det A = ad - bc.$$

The theorem says that a 2×2 matrix A is invertible if and only if $\det A \neq 0$.

Example 2.2.7. Find the inverse of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.

Theorem 2.2.8. *If* A *is an invertible* $n \times n$ *matrix, then for each* $\mathbf{b}$ *in* $\mathbb{R}^n$, *the equation* $A\mathbf{x} = \mathbf{b}$ *has the unique solution* $\mathbf{x} = A^{-1}\mathbf{b}$.

Example 2.2.9. Solve the system

$$\begin{aligned} x_1 + 2x_2 &= 1 \\ 3x_1 + 4x_2 &= 2 \end{aligned}$$

Definition 2.2.10. An **elementary matrix** is one that is obtained by performing a single elementary row operation on an identity matrix.

Example 2.2.11. $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$.
Find the products E_1A, E_2A, and E_3A, and describe how these products can be obtained by elementary row operations on A. Find an elementary matrix E such that

$$EA = \begin{bmatrix} a & b & c \\ d - 2a & e - 2b & f - 2c \\ g & h & i \end{bmatrix}.$$

Observation 2.2.12. If an elementary row operation is performed on an $m \times n$ matrix A, the resulting matrix can be written as EA, where the $m \times m$ matrix E is created by performing the same row operation on I_m.

Observation 2.2.13. Each elementary matrix E is invertible. The inverse of E is the elementary matrix of the same type that transforms E back into I.

Example 2.2.14. Find the inverses of $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

Theorem 2.2.15. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.*

Procedure 2.2.16. To find A^{-1}, row reduce the augmented matrix $\begin{bmatrix} A & I \end{bmatrix}$. If A is row equivalent to I, then $\begin{bmatrix} A & I \end{bmatrix}$ is row equivalent to $\begin{bmatrix} I & A^{-1} \end{bmatrix}$. Otherwise, A does not have an inverse.

Example 2.2.17. Find the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$.

2.2.1 Additional Thoughts and Problems

2.3 Characterizations of Invertible Matrices

2.3 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 1, 3, 11, 13, 15, 28.
- Reccommended: 5, 8, 17, 26, 35, 40 (challenge).

Theorem 2.3.1 (The Invertible Matrix Theorem)**.** *Let A be a square $n \times n$ matrix. Then the following statements are equivalent (i.e. they're either all true or all false).*

(a) A is an invertible matrix.

(b) There is an $n \times n$ matrix C such that $CA = I$.

(c) There is an $n \times n$ matrix D such that $AD = I$.

(d) A is row equivalent to I_n.

(e) A^T is an invertible matrix.

(f) A has n pivot positions.

(g) $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

(h) $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^n$.

(i) The columns of A span $\mathbb{R}^n$

(j) The columns of A are linearly independent.

(k) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.

(l) The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.

Remark 2.3.2. Note that the invertible matrix theorem only applies to *square* matrices.

Example 2.3.3. Use the Invertible Matrix Theorem to decide if A or B are invertible:

$$A = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 9 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 & -2 \\ 3 & 1 & -2 \\ -5 & -1 & 6 \end{bmatrix}$$

Definition 2.3.4. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **invertible** if there exists function $S : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$S(T(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$
$$T(S(\mathbf{x})) = \mathbf{x} \text{ for all } \mathbf{x} \text{ in } \mathbb{R}^n$$

The next theorem shows that if S exists, it's unique. We call S the **inverse** of T, written as T^{-1}.

Theorem 2.3.5. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation with standard matrix A. Then T is invertible if and only if A is invertible. In that case, the linear transformation $S(\mathbf{x}) = A^{-1}\mathbf{x}$ is the unique function satisfying the equations in the definition above.*

Example 2.3.6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation with standard matrix $A = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Describe T geometrically, and find T^{-1} if it exists.