2.5 Matrix Factorizations

McDonald Fall 2018, MATH 2210Q, 2.5 Slides

2.5 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: *nothing is due but you should definitely practice*
- Recommended: 1, 2, 3, 4, 5, 6, 9, 10, 13, 14, 15, 16.

Definition 2.5.1. A matrix with zeros below the main diagonal is called **upper triangular**. A matrix with zeros above the main diagonal is called **lower triangular**.

Suppose $A = LU$ where L is lower triangular, and U is upper triangular. Then the equation $A\mathbf{x} = \mathbf{b}$ can be written $LU\mathbf{x} = L(U\mathbf{x}) = \mathbf{b}$. Writing $\mathbf{y} = U\mathbf{x}$, we can find $\mathbf{x}$ by solving the *pair* of equations

$$L\mathbf{y} = \mathbf{b} \qquad\qquad U\mathbf{x} = \mathbf{y}$$

First solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$, and then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$.

Example 2.5.2. Suppose $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 2 & -5 & 1 & 0 \\ -3 & 8 & 3 & 1 \end{bmatrix} \begin{bmatrix} 3 & -7 & -2 & 2 \\ 0 & -2 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix}$.

Use this factorization of A to solve $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (-9, 5, 7, 11)$.

Remark 2.5.3. This factorization is useful for solving equations with the same coefficient matrix:

$$A\mathbf{x} = \mathbf{b}_1,\ A\mathbf{x} = \mathbf{b}_2,\ \ldots,\ A\mathbf{x} = \mathbf{b}_p$$

If we find a factorization when solving $A\mathbf{x} = \mathbf{b}_1$, we can use it to solve the remaining equations.

Definition 2.5.4. Let A be an $m \times n$ matrix that can be reduced to echelon form *without row interchanges*. Then A can be written in the form $A = LU$ where L is an $m \times m$ lower triangular matrix with ones on the diagonal, and U is an $m \times n$ upper triangular matrix. This factorization is called an **LU factorization**. The matrix L is invertible and called a *unit* lower triangular matrix.

Suppose that A can be reduced to echelon form U using only row replacements that add multiples of one row to another row *below* it. In this case, there are unit lower triangular elementary matrices $E_1, \ldots, E_p$ such that $E_p \cdots E_2 E_1 A = U$. Then $A = (E_p \cdots E_1)^{-1} U = LU$, where $L = (E_p \cdots E_1)^{-1}$.

Procedure 2.5.5 (Algorithm for an LU factorization)**.**

1. Reduce A to echelon form U by a sequence of row replacements.
2. Place entries in L such that the *same sequence* of row replacements reduces L to I.

Example 2.5.6. Find an LU factorization of $A = \begin{bmatrix} 3 & -7 & -2 & 2 \\ -3 & 5 & 1 & 0 \\ 6 & -4 & 0 & -5 \\ -9 & 5 & -5 & 12 \end{bmatrix}$.

Example 2.5.7. Find an LU factorization of $A = \begin{bmatrix} 2 & 4 & -1 & 5 & -2 \\ -4 & -5 & 3 & -8 & 1 \\ 2 & -5 & -4 & 1 & 8 \\ -6 & 0 & 7 & -3 & 1 \end{bmatrix}$.

Example 2.5.8. Find an LU factorization of $A = \begin{bmatrix} 2 & -4 & -2 & 3 \\ 6 & -9 & -5 & 8 \\ 2 & -7 & -3 & 9 \\ 4 & -2 & -2 & -1 \\ -6 & 3 & 3 & 4 \end{bmatrix}$.

Example 2.5.9. Let $A = \begin{bmatrix} 4 & 3 & -5 \\ -4 & -5 & 7 \\ 8 & 6 & -8 \end{bmatrix}$, $\mathbf{b}_1 = \begin{bmatrix} 2 \\ -4 \\ 6 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$. Solve $A\mathbf{x} = \mathbf{b}_1$ and $A\mathbf{x} = \mathbf{b}_2$.

2.5.1 Additional Thoughts and Problems