NOTE: These slides contain *both* Sections 3.1 and 3.2.

3.3 Cramer's Rule, Volume, and Linear Transformations

McDonald Fall 2018, MATH 2210Q, 3.3 Slides

3.3 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: TBD.
- Recommended: TBD.

3.3.1 Cramer's rule

Theorem 3.3.1 (Cramer's Rule)**.** *Let A be an invertible $n \times n$ matrix. For any $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution $\mathbf{x}$ of $A\mathbf{x} = \mathbf{b}$ has entries given by*

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \qquad i = 1, 2, \ldots, n$$

where

$$A_i(\mathbf{b}) = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_{i-1} & \mathbf{b} & \mathbf{a}_{i+1} & \cdots & \mathbf{a}_n \end{bmatrix}$$

Example 3.3.2. Use Cramer's rule to solve the system

$$\left\{ \begin{array}{rcrcr} 3x_1 & - & 2x_2 & = & 6 \\ -5x_1 & + & 4x_2 & = & 8 \end{array} \right.$$

Example 3.3.3. Consider the following system of equations where $a \neq 0$. Prove that if $a \neq 24$ then the system has exactly one solution. What does the solution set look like?

$$\left\{ \begin{array}{rcrcr} ax & - & 6y & = & -1 \\ 4x & - & y & = & 3 \end{array} \right.$$

Definition 3.3.4. The **adjugate** (or **classical adjoint**) of A, is

$$\operatorname{adj}(A) = \begin{bmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{bmatrix}^T = \begin{bmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & \vdots & & \vdots \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{bmatrix}.$$

Where $C_{ij} = (-1)^{i+j} \det A_{ij}$.

Theorem 3.3.5 (Inverse Formula)**.** *Let A be an invertible $n \times n$ matrix. Then*

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Example 3.3.6. Find the inverse of the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$$

Remark 3.3.7. This formula for the inverse is really useful for theoretical calculations, but in almost all cases, our algorithm of reducing to the identity is ***much*** more efficient.

3.3.2 Determinants as Area or Volume

Theorem 3.3.8. *If A is a 2×2 matrix, then the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, then the volume of the parallelepiped determined by the columns of A is $|\det A|$.*

Example 3.3.9. Calculate the area of the parallelogram with vertices $(0, 0)$, $(1, 2)$, $(2, 3)$ and $(3, 5)$.

Example 3.3.10. Find the area of the parallelogram with vertices $(-2, -2)$, $(0, 3)$, $(4, -1)$ and $(6, 4)$.

Example 3.3.11. Find the area of the parallelepiped with one vertex at the origin, and adjacent vertices $(1, 0, -1)$, $(4, 5, 6)$, $(7, 3, 9)$.

Example 3.3.12. Find the area of the triangle with vertices $(1, 2)$, $(4, 3)$, and $(3, 5)$.

3.3.3 Linear Transformations

Theorem 3.3.13. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation determined by a 2×2 matrix A. If S is a parallelogram in $\mathbb{R}^2$, then*

$$\{\textit{area of } T(S)\} = |\det A| \cdot \{\textit{area of } S\}$$

If $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation determined by a 3×3 matrix A, and S is a parallelepiped in $\mathbb{R}^3$, then

$$\{\textit{volume of } T(S)\} = |\det A| \cdot \{\textit{volume of } S\}$$

Let S be the parallelogram determined by the vectors $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{b}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation with standard matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix}$. Find the area of $T(S)$.

3.3.4 Additional Notes and Problems