4.1 Vector Spaces and Subspaces

McDonald Fall 2018, MATH 2210Q, 4.1 Slides

4.1 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 1, 3, 8, 13, 23, 31.
- Recommended: 12, 15, 17, 22, 32.

A lot of the theory in Chapters 1 and 2 used simple and obvious algebraic properties of $\mathbb{R}^n$, which we discussed in Section 1.3. Many other mathematical systems have the same properties. The properties we are interested in are listed in the following definition.

Definition 4.1.1. A **vector space** is a nonempty set V of objects, called *vectors*, on which two operations are defined: *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms below. The axioms must hold for all $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in V, and all scalars c and d.

1. The sum of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u}+\mathbf{v}$, is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There is a **zero** vector, $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each $\mathbf{u}$ in V, there is a vector $-\mathbf{u}$ in V such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. The scalar multiple of $\mathbf{u}$ by c, denoted $c\mathbf{u}$, is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$.

Example 4.1.2. Show the set of all $m\times n$ matrices with entries in $\mathbb{R}$, denoted $M_{m\times n}$, with addition and scalar multiplication defined using addition and scalar multiplication of matrices is a vector space.

The following properties of vector spaces are also useful.

Proposition 4.1.3. *For each* $\mathbf{u}$ *in* V *and scalar* c*,*

$$0\mathbf{u} = \mathbf{0}$$
$$c\mathbf{0} = \mathbf{0}$$
$$-\mathbf{u} = (-1)\mathbf{u}$$

Example 4.1.4. Show that the set $\mathbb{P}_n$ of all polynomials of degree at most n with addition by combining like coefficients, and multiplication by c by scaling each coefficient by c is a vector space.

Example 4.1.5. Show that the set W of all real-valued functions on $\mathbb{R}$ with addition given by $f + g = (f + g)(x)$ for all $f, g \in W$, and scalar multiplication given by $cf = cf(x)$ is a vector space.

Definition 4.1.6. A **subspace** of a vector space is a subset H of V that has the properties:

(a) The zero vector of V is in H.

(b) H is closed under vector addition: for every $\mathbf{u}$ and $\mathbf{v}$ in H, the sum $\mathbf{u}+\mathbf{v}$ is in H.

(c) H is closed under scalar multiplication: for all $\mathbf{u}$ in H and scalar c, the vector $c\mathbf{u}$ is in H.

Example 4.1.7. The set $\{\mathbf{0}\}$ is a subspace of any vector space, called the **zero subspace**.

Example 4.1.8. Let $H = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$. Show that H is a subspace of $M_{2\times 2}$.

Example 4.1.9. Show that $H = \{f : \mathbb{R} \to \mathbb{R} : f \text{ is differentiable}\}$ is a subspace of W, from 4.1.5.

Example 4.1.10. Given $\mathbf{v}_1$ and $\mathbf{v}_2$ in a vector space V. Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ is a subspace of V.

Theorem 4.1.11. *If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are in a vector space V, then Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ is a subspace of V.*

Definition 4.1.12. We call $\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ the **subspace spanned** (or **generated**) by $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. For any subspace H of V, a **spanning set** (or **generating set**) for H is a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example 4.1.13. Let

$$H = \{(a - 3b, b - a, a, b) : a \text{ and } b \text{ in } \mathbb{R}\}.$$

Show that H is a subspace of $\mathbb{R}^4$.

Example 4.1.14. Show $H = \{at^2 + at + a : a \text{ in } \mathbb{R}\}$ is a subspace of $\mathbb{P}_n$.

Example 4.1.15. Show that $\mathbb{D} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x^2 + y^2 < 1 \right\}$ is not a subspace of $\mathbb{R}^2$.

Example 4.1.16. Let V be the set of all arrows in three dimensional space with two arrows considered as "equal" if they point in the same direction and have the same length. Define addition using the parallelogram rule, and define $c\mathbf{v}$ as the vector whose length is $|c|$ times the length of $\mathbf{v}$, in the same direction as $\mathbf{v}$ if $c \geq 0$, and the opposite direction if $c < 0$. Show that V is a vector space.

4.1.1 Additional Notes and Problems