4.2 Null Spaces Column Spaces and linear Transformations

McDonald Fall 2018, MATH 2210Q, 4.2 Slides

4.2 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: TBD
- Recommended: TBD

Definition 4.2.1. The **null space** of an $m \times n$ matrix A, written as $\text{Nul}\,A$, is the set of all solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$. In set notation,

$$\text{Nul}\,A = \{\mathbf{x} : \mathbf{x} \text{ is in } \mathbb{R}^n \text{ and } A\mathbf{x} = \mathbf{0}\}.$$

Example 4.2.2. Let $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$, and $\mathbf{u} = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$. Show that $\mathbf{u}$ is in $\text{Nul}\,A$.

Theorem 4.2.3. *The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\mathbf{x} = \mathbf{0}$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.*

Example 4.2.4. Let H be the set of all vectors in $\mathbb{R}^4$ whose coordinates satisfy the equations $x_1 - 2x_2 + 5x_3 = x_4$ and $x_3 - x_1 = x_2$. Show that H is a subspace of $\mathbb{R}^4$.

Example 4.2.5. Find a spanning set for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}.$$

Remark 4.2.6. These points will be useful later on:

1. The method in Example 4.2.14 gives a spanning set that's automatically linearly independent.
2. When $\operatorname{Nul} A$ contains nonzero vectors, the number of vectors in the spanning set of $\operatorname{Nul} A$ equals the number of free variables in the equation $A\mathbf{x} = \mathbf{0}$.

Definition 4.2.7. The **column space** of an $m \times n$ matrix A, written as $\operatorname{Col} A$, is the set of all linear combinations of the columns of A. If $A = [\ \mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n\ ]$, then

$$\operatorname{Col} A = \operatorname{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Remark 4.2.8. A typical vector in $\operatorname{Col} A$ can be written as $A\mathbf{x}$ for some $\mathbf{x}$, since the notation $A\mathbf{x}$ stands for a linear combination of the columns of A. In other words

$$\operatorname{Col} A = \{\mathbf{b} : \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \text{ in } \mathbb{R}^n\}$$

Example 4.2.9. Find a matrix A such that $W = \operatorname{Col} A$.

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$$

Theorem 4.2.10. *The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.*

Theorem 4.2.11. *The column space of an $m \times n$ matrix A is all of $\mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for each $\mathbf{b}$ in $\mathbb{R}^m$.*

How are the null space and column space of a matrix related? In the next example, we'll see that the two spaces are very different. If you're interested, Section 4.6 reveals some surprising connections.

Example 4.2.12. Consider the following matrix.

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}.$$

(a) If the column space of A is a subspace of $\mathbb{R}^k$, what is k?
(b) If the null space of A is a subspace of $\mathbb{R}^k$, what is k?
(c) Find a nonzero vector in $\operatorname{Col} A$, and a nonzero vector in $\operatorname{Nul} A$.
(d) Is $\mathbf{u} = (3, -2, -1, 0)$ in $\operatorname{Nul} A$? Could it be in $\operatorname{Col} A$?
(e) Is $\mathbf{v} = (3, -1, 3)$ in $\operatorname{Col} A$? Could it be in $\operatorname{Nul} A$?

Contrast Between Nul A and Col A for an $m \times n$ Matrix A

Nul A	Col A
1. Nul A is a subspace of $\mathbb{R}^n$.	**1.** Col A is a subspace of $\mathbb{R}^m$.
2. Nul A is implicitly defined; that is, you are given only a condition ($A\mathbf{x} = \mathbf{0}$) that vectors in Nul A must satisfy.	**2.** Col A is explicitly defined; that is, you are told how to build vectors in Col A.
3. It takes time to find vectors in Nul A. Row operations on $[\,A \quad \mathbf{0}\,]$ are required.	**3.** It is easy to find vectors in Col A. The columns of A are displayed; others are formed from them.
4. There is no obvious relation between Nul A and the entries in A.	**4.** There is an obvious relation between Col A and the entries in A, since each column of A is in Col A.
5. A typical vector $\mathbf{v}$ in Nul A has the property that $A\mathbf{v} = \mathbf{0}$.	**5.** A typical vector $\mathbf{v}$ in Col A has the property that the equation $A\mathbf{x} = \mathbf{v}$ is consistent.
6. Given a specific vector $\mathbf{v}$, it is easy to tell if $\mathbf{v}$ is in Nul A. Just compute $A\mathbf{v}$.	**6.** Given a specific vector $\mathbf{v}$, it may take time to tell if $\mathbf{v}$ is in Col A. Row operations on $[\,A \quad \mathbf{v}\,]$ are required.
7. Nul $A = \{\mathbf{0}\}$ if and only if the equation $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.	**7.** Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.
8. Nul $A = \{\mathbf{0}\}$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.	**8.** Col $A = \mathbb{R}^m$ if and only if the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

Definition 4.2.13. A **linear transformation** T from a vector space V into a vector space W is a rule that assigns to each vector $\mathbf{x}$ in V one and only one vector $T(\mathbf{x})$ in W such that for all $\mathbf{u}, \mathbf{v}$ in V and real number c,

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$

The **kernel** (or **null space**) of T is the set of all $\mathbf{u}$ in V such that $T(\mathbf{u}) = \mathbf{0}$. The **range** of T is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V.

Example 4.2.14. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be given by $T(\mathbf{x}) = A\mathbf{x}$. What are the kernel and range of T?

$$A = \begin{bmatrix} -1 & -5 & 7 \\ 2 & 7 & -8 \end{bmatrix}.$$

Example 4.2.15. Let V be the space of all differentiable functions whose derivatives are continuous, and W be the space of all continuous functions. Show that $D : V \to W$ by $f \mapsto f'$ is a linear transformation. What is the kernel of D? What is the range?