4.3 Linearly Independent Sets

McDonald Fall 2018, MATH 2210Q, 4.3 Slides

4.3 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 3, 4, 14, 21, 29, 30
- Recommended: 8, 10, 15, 23, 24, 31

Definition 4.3.1. An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors in a vector space V is called **linearly independent** if the vector equation

$$c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \qquad (\star)$$

has *only* the trivial solution $c_1 = 0, \ldots, c_p = 0$. The set S is called **linearly dependent** if there are $c_1, \ldots, c_p$ *not all* zero, such that $(\star)$ holds. In this case, $(\star)$ is called a **linear dependence relation**.

Theorem 4.3.2. *An indexed set $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ of two or more vectors, with $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.*

Example 4.3.3. Let $\mathbf{p}_1(t) = 1$, $\mathbf{p}_2(t) = t^2$, $\mathbf{p}_3(t) = 4 - t^2$ in $\mathbb{P}_2$. Is $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ linearly independent?

Example 4.3.4. Let $C[0,1]$ be the space of real-valued continuous functions on $0 \leq t \leq 1$. Is $\{\sin^2 t, \cos^2 t\}$ linearly independent? Is $\{1, \sin^2 t, \cos^2 t\}$?

Definition 4.3.5. Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_p\}$ in V is a **basis** for H if

(a) $\mathcal{B}$ is a linearly independent set, and

(b) $\mathcal{B}$ spans all of H; that is,

$$H = \text{Span}(\mathcal{B}) = \text{Span}\{\mathbf{b}_1, \dots, \mathbf{b}_p\}$$

Remark 4.3.6. Since $H = V$ is a subspace of V, we can also talk about a basis for V.

Example 4.3.7. Let A be an invertible $n \times n$ matrix, and $\mathcal{B} = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$. Is $\mathcal{B}$ a basis for $\mathbb{R}^n$?

Example 4.3.8. Let $\mathcal{B} = \{\mathbf{e}_1, \cdots, \mathbf{e}_n\}$ be the columns of the $n \times n$ identity matrix I. Show that $\mathcal{B}$ is a basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$.

Example 4.3.9. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Is $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ a basis for $\mathbb{R}^3$?

Example 4.3.10. Verify $\mathcal{B} = \{1, t, t^2, \cdots, t^n\}$ is a basis for $\mathbb{P}_n$. This is the **standard basis** for $\mathbb{P}_n$.

4.3.1 The spanning set theorem

Example 4.3.11. Let $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 6 \\ 16 \\ -5 \end{bmatrix}$, and $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. Verify that $\mathbf{v}_3 = 5\mathbf{v}_1 + 3\mathbf{v}_2$, and $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\} = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$. What is a basis for H?

Definition 4.3.12. Let $S = \{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$ be a set in V, and let $H = \text{Span}\{\mathbf{v}_1, \cdots, \mathbf{v}_p\}$.

(a) If one of the vectors in S, say $\mathbf{v}_k$, is a linear combination of the remaining vectors in S, then the set formed by removing $\mathbf{v}_k$ from S still spans H.

(b) If $H \neq \{\mathbf{0}\}$, some subset of S is a basis for H.

4.3.2 Bases for $\operatorname{Col} A$ and $\operatorname{Nul} A$

Example 4.3.13. Find a basis for $\operatorname{Col} U$, where $U = \begin{bmatrix} \mathbf{u}_1 & \cdots & \mathbf{u}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Example 4.3.14. Below, A is row equivalent to U from the last example. Find a basis for $\operatorname{Col} A$.

$$A = \begin{bmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_5 \end{bmatrix} = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

Theorem 4.3.15. *The pivot columns of a matrix A form basis for* $\operatorname{Col} A$.

Watchout! 4.3.16. We need to reduce A to echelon form U to find pivot columns. However, the pivot columns of U ***do not*** form a basis for $\operatorname{Col} A$. You have to use the pivot columns of A.

Example 4.3.17. Find a basis for Nul A, where A is the same as the previous example:

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

4.3.3 Two views of a basis

Example 4.3.18. Which of the following is a basis for $\mathbb{R}^3$?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix} \right\}$$

Remark 4.3.19. In one sense, a basis for V is a spanning set of V that is as small as possible. In another sense, a basis for V is a linearly independent set that is as large as possible.

4.3.4 Additional Notes and Problems