4.4 Coordinate Systems

McDonald Fall 2018, MATH 2210Q, 4.4 Slides

4.4 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 2, 5, 10, 13, 15, 17
- Recommended: 3, 7, 11, 21, 23, 32

An important reason for specifying a basis $\mathcal{B}$ for a vector space V is to give V a "coordinate system." We will show that if $\mathcal{B}$ contains n vectors, then the coordinate system makes V look like $\mathbb{R}^n$.

Theorem 4.4.1. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each $\mathbf{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that*

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n.$$

Definition 4.4.2. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for V, and $\mathbf{x}$ is in V. The **coordinates of x relative to** $\mathcal{B}$ (or the $\mathcal{B}$**-coordinates of x**) are the weights $c_1, \ldots, c_n$ such that $\mathbf{x} = c_1\mathbf{b}_1 + \cdots + c_p\mathbf{b}_n$. The vector in $\mathbb{R}^n$

$$\left[\mathbf{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

is called the **coordinate vector of x** (relative to $\mathcal{B}$), and the mapping from V to $\mathbb{R}^n$ by $\mathbf{x} \mapsto \left[\mathbf{x}\right]_{\mathcal{B}}$ is called the **coordinate mapping** (determined by $\mathcal{B}$).

Example 4.4.3. Consider the basis $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\}$ of $\mathbb{R}^2$. Suppose $\mathbf{x}$ has the coordinate vector $\left[\mathbf{x}\right]_{\mathcal{B}} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find $\mathbf{x}$.

Example 4.4.4. Consider the **standard basis** for $\mathbb{R}^2$, $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2\}$. Let $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$. Find $\left[\mathbf{x}\right]_{\mathcal{B}}$.

Example 4.4.5. In the previous two examples, we considered the coordinates of $\mathbf{x} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ in $\mathbb{R}^2$ relative to the bases $\mathcal{B}$ and $\mathcal{E}$. Interpret these examples graphically.

Once we fix a basis $\mathcal{B}$ for $\mathbb{R}^n$, the $\mathcal{B}$-coordinates of a specified $\mathbf{x}$ are easy to find:

Example 4.4.6. Let $\mathcal{B} = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ be a basis for $\mathbb{R}^2$. Find the coordinate vector of $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

The matrix we used in the previous example changed the $\mathcal{B}$-coordinates of a vector $\mathbf{x}$ into the standard coordinates for $\mathbf{x}$. We can generalize this to $\mathbb{R}^n$:

Definition 4.4.7. Suppose $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for $\mathbb{R}^n$, and define the matrix $P_{\mathcal{B}} = \begin{bmatrix} \mathbf{b}_1 & \cdots & \mathbf{b}_n \end{bmatrix}$. Then the vector equation $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$ is equivalent to

$$\mathbf{x} = P_{\mathcal{B}}\big[\,\mathbf{x}\,\big]_{\mathcal{B}}.$$

$P_{\mathcal{B}}$ is called the **change-of-coordinates matrix** from $\mathcal{B}$ to the standard basis for $\mathbb{R}^n$.

Theorem 4.4.8. *Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then the mapping $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ is a one-to-one and onto linear transformation from V to $\mathbb{R}^n$.*

Remark 4.4.9. A one-to-one linear transformation between a vector space V onto another vector space W is called an ***isomorphism***, from the Greek words *iso* meaning "the same," and *morph* meaning "structure." The map $\mathbf{x} \mapsto [\,\mathbf{x}\,]_{\mathcal{B}}$ gives us a way to view V as indistinguishable from $\mathbb{R}^n$.

Example 4.4.10. Let $\mathcal{B} = \{1, t, t^2\}$ be the standard basis of $\mathbb{P}_2$. Let $\mathbf{p}_0 = a_0 + a_1 t + a_2 t^2$, $\mathbf{p}_1 = t^2$, $\mathbf{p}_2 = 4 + t + 5t^2$ and $\mathbf{p}_3 = 3 + 2t$.

(a) Find $[\,\mathbf{p}\,]_{\mathcal{B}}$ for $\mathbf{p} = \mathbf{p}_0, \ldots, \mathbf{p}_3$.

(b) Use coordinate vectors to show that $\mathbf{p}_1$, $\mathbf{p}_2$, and $\mathbf{p}_3$ are linearly independent.

Remark 4.4.11. In this example, $\mathbb{P}_2$ is *isomorphic* to $\mathbb{R}^3$. In general, $\mathbb{P}_n$ is isomorphic to $\mathbb{R}^{n+1}$.

Example 4.4.12. Let $\mathbf{v}_1 = \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{x} = \begin{bmatrix} 3 \\ 12 \\ 7 \end{bmatrix}$. Suppose $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

(a) Find a basis $\mathcal{B}$ for H.

(b) Show that $\left[\mathbf{x}\right]_{\mathcal{B}}$ is a map from H to $\mathbb{R}^2$, hence, an *isomorphism* between H and $\mathbb{R}^2$.

(c) Show $\mathbf{x}$ is in H, and find the coordinate vector of $\mathbf{x}$ relative to $\mathcal{B}$.

Remark 4.4.13. This example shows that $\mathbf{v}_1, \mathbf{v}_2$ span a plane in $\mathbb{R}^3$ that is *isomorphic* to $\mathbb{R}^2$. In fact, if $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a linearly independent set of vectors in $\mathbb{R}^m$, then $H = \text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is isomorphic to $\mathbb{R}^n$ under the map $\mathbf{x} \mapsto \left[\mathbf{x}\right]_{\mathcal{B}}$ where $\mathcal{B} = S$.

Definition 4.4.14. If V is spanned by a finite set, then the **dimension** of V is the number of vectors in a basis for V. The dimension of the zero space, $\{\mathbf{0}\}$ is defined to be zero. If V is not spanned by a finite set, then V is said to be **infinite-dimensional**.