NOTE: These slides contain *both* Section 5.1 and 5.2.

5.1 Eigenvectors and Eigenvalues

McDonald Fall 2018, MATH 2210Q, 5.1 Slides & 5.2

5.1 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 2, 6, 7, 13, 21, 23, 24
- Recommended: 11, 15, 19, 25, 27, 31

Example 5.1.1. Let $A = \begin{bmatrix} 3 & -2 \\ 1 & 0 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Compute $A\mathbf{u}$ and $A\mathbf{v}$.

Remark 5.1.2. In this example, it turns out $A\mathbf{v}$ is just $2\mathbf{v}$, so A only stretches $\mathbf{v}$.

Definition 5.1.3. An **eigenvector** of an $n \times n$ matrix is a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \lambda\mathbf{v}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution $\mathbf{x} = \mathbf{v}$ of the equation $A\mathbf{x} = \lambda\mathbf{x}$; such a $\mathbf{v}$ is called an *eigenvector corresponding to* λ.

Example 5.1.4. Let $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 6 \\ -5 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

(a) Are $\mathbf{u}$ and $\mathbf{v}$ eigenvectors of A?

(b) Show that 7 is an eigenvalue of $A = \begin{bmatrix} 1 & 6 \\ 5 & 2 \end{bmatrix}$.

Procedure 5.1.5 (Determining if λ is an eigenvalue)**.** The scalar λ is an eigenvalue for a matrix A if and only if the equation

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

has a nontrivial solution. Just reduce the associated augmented matrix!

Definition 5.1.6. The set of all solutions to $A\mathbf{x} = \lambda\mathbf{x}$ is the nullspace of the matrix $A - \lambda I$, and therefore is a subspace of $\mathbb{R}^n$. We call this the **eigenspace** of A corresponding to λ.

Remark 5.1.7. Even though we used row reduction to find eigen*vectors*, we cannot use it to find eigen*values*. An echelon for a matrix A doesn't usually have the same eigenvalues as A.

Example 5.1.8. Let $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find a basis for the eigenspace corresponding to $\lambda = 2$.

Theorem 5.1.9. *The eigenvalues of a triangular matrix are the entries on its main diagonal.*

Example 5.1.10. Let $A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & 3 & 4 \end{bmatrix}$. What are the eigenvalues of A and B? What does it mean for A to have an eigenvalue of 0?

Theorem 5.1.11. *If* $\mathbf{v}_1, \ldots, \mathbf{v}_r$ *are eigenvectors that correspond to distinct eigenvalues* $\lambda_1, \ldots, \lambda_r$ *of an* $n \times n$ *matrix* A*, then the set* $\{\mathbf{v}_1, \ldots, \mathbf{v}_r\}$ *is linearly independent.*

Example 5.1.12. Let $C = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. Find the eigenspaces corresponding to $\lambda = 0, 1$.

Remark 5.1.13. Note, the matrix C is RREF form for A, but the eigenvalues are different.

Additional Notes/Problems

In the next section, we'll be using determinants to find eigenvalues of a matrix. We'll close this section by reviewing some of the properties we know for determinants.

Proposition 5.1.14. *Suppose A is an $n \times n$ matrix that can be reduced to echelon form U using only row replacements and r row interchanges. Then the determinant of A is*

$$\det A = (-1)^r \cdot u_{11} u_{22} \cdots u_{nn}.$$

Proposition 5.1.15. *Let A and B be $n \times n$ matrices.*

(a) *A is invertible if and only if $\det A \neq 0$.*
(b) $\det AB = (\det A)(\det B)$.
(c) $\det A^T = \det A$.
(d) *If A is triangular,* $\det A = a_{11} a_{22} \cdots a_{nn}$.
(e) *A row replacement does not change the determinant. A row interchange changes the sign of the determinant. Scaling a row scales the determinant by the same factor.*

We also recall the invertible matrix theorem.

Theorem 5.1.16 (The Invertible Matrix Theorem)**.** *Let A be a square $n \times n$ matrix. Then the following statements are equivalent (i.e. they're either all true or all false).*

(a) *A is an invertible matrix.*
(b) *There is an $n \times n$ matrix C such that $CA = I$.*
(c) *There is an $n \times n$ matrix D such that $AD = I$.*
(d) *A is row equivalent to I_n.*
(e) *A^T is an invertible matrix.*
(f) *A has n pivot positions.*
(g) *$A\mathbf{x} = \mathbf{0}$ has only the trivial solution.*
(h) *$A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ in $\mathbb{R}^n$.*
(i) *The columns of A span $\mathbb{R}^n$*
(j) *The columns of A are linearly independent.*
(k) *The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is one-to-one.*
(l) *The transformation $\mathbf{x} \mapsto A\mathbf{x}$ is onto.*

We can also add the following to the list:

(m) *The determinant of A is not zero.*
(n) *The number 0 is not an eigenvalue of A*

5.2 The Characteristic Equation (finding eigenvalues)

McDonald Fall 2018, MATH 2210Q, 5.2 Slides

5.2 Homework: Read section and do the reading quiz. Start with practice problems.

- ***Hand in***: 2, 5, 9, 12, 15, 21
- Recommended: 19, 20

Example 5.2.1. Find the eigenvalues of $A = \begin{bmatrix} 2 & 3 \\ 3 & -6 \end{bmatrix}$.

Definition 5.2.2. The equation $\det(A - \lambda I) = 0$ is called the **characteristic equation** of A.

Proposition 5.2.3. *A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the characteristic equation*

$$\det(A - \lambda I) = 0.$$

Example 5.2.4. Find the characteristic equation and eigenvalues of $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Definition 5.2.5. If A is an $n \times n$ matrix, then $\det(A - \lambda I)$ is a polynomial of degree n called the **characteristic polynomial** of A. The **multiplicity** of an eigenvalue λ is its multiplicity as a root of the characteristic polynomial.

Example 5.2.6. The characteristic polynomial of a 6×6 matrix A is $\lambda^6 - 4\lambda^5 - 12\lambda^4$. Find the eigenvalues of A and their multiplicities.

Example 5.2.7. Find the eigenvalues and bases for the corresponding eigenspaces of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 1 \\ 0 & -1 & 4 \end{bmatrix}.$$

Additional Notes/Problems