Derivatives of $\sin x$, $\cos x$, $\tan x$ and $\sec x$.

Goals

- derivatives of sine and cosine
- derivatives of tangent and secant
- general derivatives involving trig functions
- applications

21.1 Investigating the Derivative of $\sin x$ graphically.

Example 21.1.1. Graph the derivative of $\sin x$ by estimating its values at

$$x = 0,\ \pi/4,\ \pi/2,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 3\pi/2,\ 7\pi/4, \text{ and } 2\pi.$$

$f(x) = \sin x$

$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin(x)}{h}$

$f'(c) \approx \frac{f(c+0.001)-f(c)}{0.001}$

$f'(0) \approx \frac{\sin(0+0.001)-\sin 0}{0.001} \approx 1$ $\qquad \frac{d}{dx}(\sin x)\Big|_{x=0} \approx 1$

$f'(\pi/4) = \frac{\sin(\pi/4+0.001)-\sin(\pi/4)}{0.001} \approx 0.7$

$f'(\pi/2) \approx 0$

$f'(3\pi/4) \approx -0.7$

$f'(\pi) \approx -1$

$f'(\frac{5\pi}{4}) \approx -0.7$

$f'(\frac{3\pi}{2}) \approx 0$

$f'(7\pi/4) \approx 0.7$

$f'(2\pi) \approx 1$

looks a lot like

$\frac{d}{dx} \sin x = \cos x$

Example 21.1.2. Graph the derivative of $\cos x$ using the applet

`https://www.geogebra.org/m/u4ydxuhc`

on Geogebra.

It looks a lot like

$$\frac{d}{dx} \cos x = -\sin x$$

21.2 Differentiating $\sin x$ and $\cos x$

Theorem 21.2.1.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

Example 21.2.2. Differentiate the following

(a) $y = \sin(x^2)$

ch rule

$$\frac{d}{dx}\sin(x^2) = \cos(x^2) * 2x$$

(b) $y = \sin^2 x$

$(\sin x)^2$

$2 \sin x \cos x$

Example 21.2.3. Differentiate the following

(a) $\tan x$ $\frac{\sin x}{\cos x}$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x}$$

$$= \boxed{\frac{1}{\cos^2 x}} = \sec^2 x$$

(b) $\sec x$ $= \frac{1}{\cos x}$

$$\frac{d}{dx}(\sec x) = \frac{d}{dx}\left(\frac{1}{\cos x}\right) = \frac{d}{dx}\left((\cos x)^{-1}\right)$$

$$= -(\cos x)^{-2}(-\sin x)$$

$$= \frac{\sin x}{\cos^2 x}$$

$$= \frac{\sin x}{\cos x} * \frac{1}{\cos x}$$

$$= \sec x \tan x$$

Theorem 21.2.4.

$$\frac{d}{dx}\sin x = \cos x \qquad \frac{d}{dx}\cos x = -\sin x$$

$$\frac{d}{dx}\tan x = \sec^2 x \qquad \frac{d}{dx}\sec x = \sec x \tan x$$

$$\frac{d}{dx}\cot x = -\csc^2 x \qquad \frac{d}{dx}\csc x = -\csc x \cot x$$

21.2.1 Extra Examples

Example 21.2.5. Differentiate the following

ELFS

(a) $3x\sin(x^2)$

$$\frac{d}{dx}\left(3x\sin(x^2)\right) = 3x*\left(2x\cos x^2\right) + 3\sin(x^2)$$

product rule.

(b) $7\cos^2(3x+5)$

$$\frac{d}{dx}\,7\cos^2(3x+5) = 7\left(2\cos(3x+5)\right)*\frac{d}{dx}\left(\cos(3x+5)\right)$$
$$= 14\cos(3x+5)\left(-\sin(3x+5)*3\right)$$

???

(c) $x\tan^2 x$

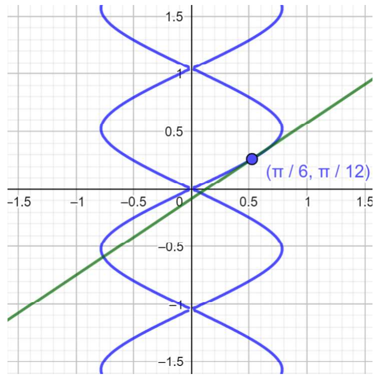
$$\frac{d}{dx}\,x\tan^2 x = \tan^2 x + 2x(\tan x)\sec^2 x$$

product rule

chain rule

21.2.2 Extra Examples (cont)

Example 21.2.6. Below is a graph of the helix $2\sin^2\left(x+\frac{\pi}{2}\right) = \cos^2(3y)+1.$



Find the slope of the tangent line at the point $(\pi/6, \pi/12)$.

(you can leave your answer in terms of trig functions and we'll simplify together)

want $\frac{dy}{dx}$

$$\frac{d}{dx}\left(2\sin^2\left(x+\frac{\pi}{2}\right)\right) = \frac{d}{dx}\left(\cos^2(3y)+1\right)$$

$$\frac{d}{dx}\left(2\left(\sin\left(x+\frac{\pi}{2}\right)\right)^2\right) = \frac{d}{dx}\left(\left(\cos(3y)\right)^2+1\right)$$

$$4\sin\left(x+\frac{\pi}{2}\right)\cdot\cos\left(x+\frac{\pi}{2}\right) = 2\cos(3y)\left(-\sin(3y)\right)3\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{4\sin\left(x+\frac{\pi}{2}\right)\cos\left(x+\frac{\pi}{2}\right)}{-6\cos(3y)\sin(3y)} = \cancel{\frac{\sqrt{3}}{2}}$$

plug in

$\frac{\sqrt{3}}{3}$?

$$\sin 2\theta = 2\sin\theta\cos\theta$$