31.2 Solutions to Differential Equations (continued)

31.2. Goals

- solving differential equations
- slope fields
- existence and uniqueness

Example 31.2.1. Determine whether each function is a solution to $y'' - y = 0$

(a) Ce^x

(b) Ce^{-x}

$$y = Ce^x \Rightarrow y'' = (Ce^x)' = Ce^x$$

$$y'' - y = Ce^x - Ce^x = 0 \checkmark$$

$$y = Ce^{-x} \Rightarrow y'' = (-Ce^{-x}) = Ce^{-x}$$

$$y'' - y = Ce^{-x} - Ce^{-x} = 0 \checkmark$$

Example 31.2.2. Determine whether each function is a solution to the equation $xy' - 2y = 0$. If either is a solution, find the particular solution whose graph passes through the point $(1, 3)$.

(a) Cx^2

(b) $x^2 + C$

a) $y = Cx^2 \Rightarrow y' = 2Cx$

$xy' - 2y = x(2Cx) - 2(Cx^2) = 0$ ✓

$(1,3) \Rightarrow y(1) = 3 \quad 3 = C(1)^2 = C$

so part. soln is $y = 3x^2$

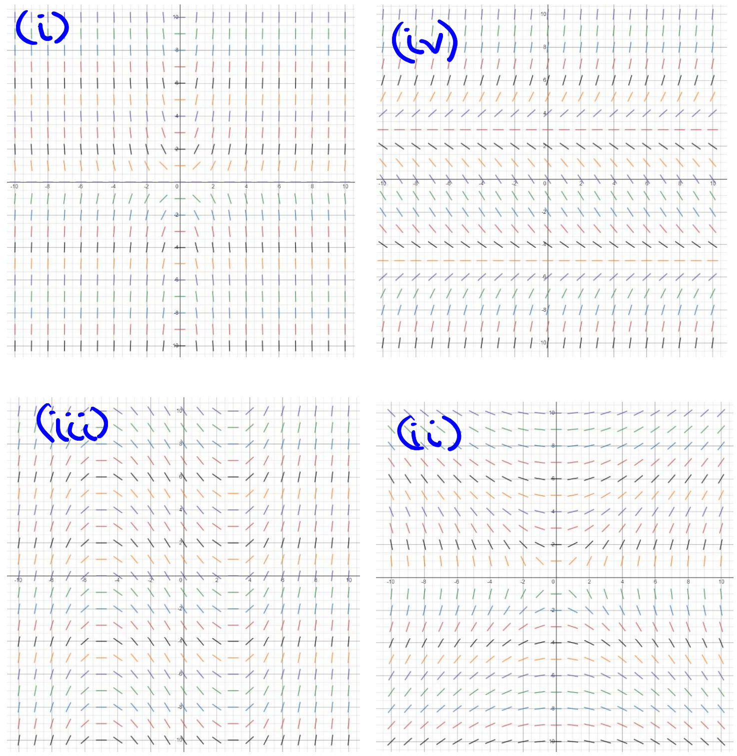
b) $y = x^2 + C \Rightarrow y' = 2x$

$xy' - 2y = x(2x) - 2(x^2 + c) = -2c \neq 0$

Example 31.2.3. .

(a) Match each differential equation to its slope field.

(b) On each graph, find the particular solution through the origin.



(i) $\frac{dy}{dx} = xy$

(ii) $\frac{dy}{dx} = x/y$

(iii) $\frac{dy}{dx} = (x-3)(x+5)$

(iv) $\frac{dy}{dx} = (y-3)(y+5)$

Example 31.3.1. Suppose a hot or cold beverage is put in a room that is kept at 65 degrees. Then the rate of change of the temperature of the beverage is

$$\frac{dT}{dt} = k(65 - T),$$

where t is a positive constant.

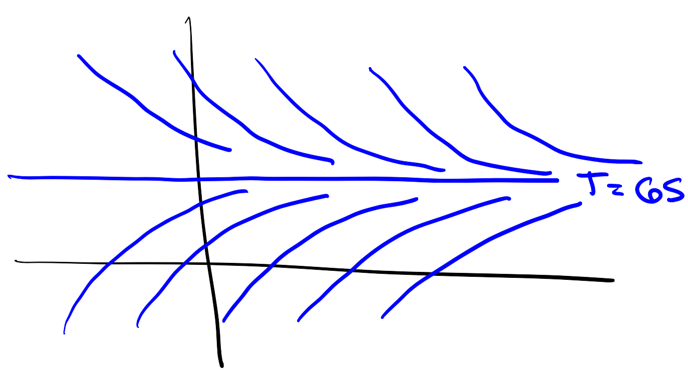
(a) What must the temperature of the beverage be in order for its temperature to remain constant.

If $T = 65$ $\frac{dT}{dt} = 0 \Rightarrow$ constant

(b) For what temperatures is the beverage cooling down?

If $T > 65 \Rightarrow \frac{dT}{dt} < 0$

(c) Sketch representative solution curves corresponding to a variety of intitial conditions.

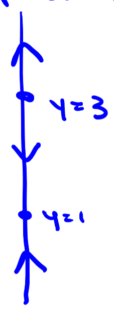


Example 31.3.3. Do a qualitative analysis of the solutions to the differential equation

$$\frac{dy}{dx} = (y-1)(y-3)$$

Sketch representatives of the family of solutions.

draw a "phase" line!

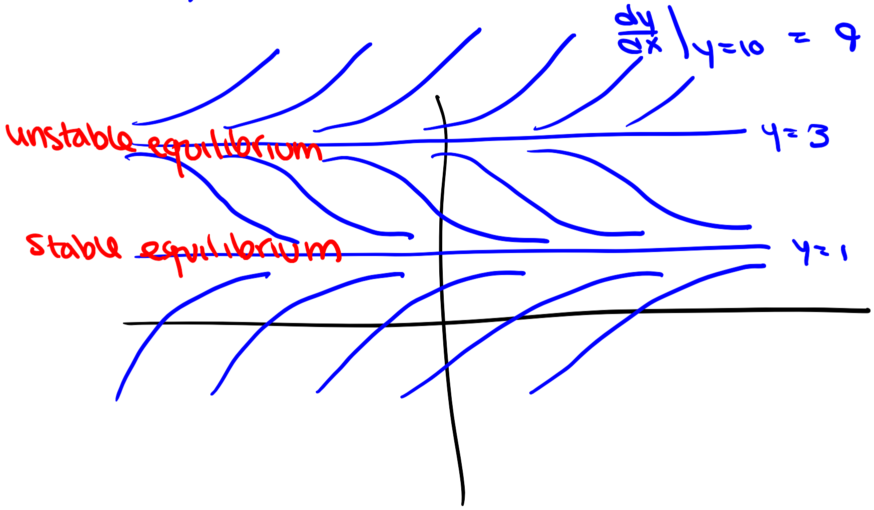


$\frac{dy}{dx} = 0 \iff y=1,\ y=3$

$\left.\frac{dy}{dx}\right|_{y=2} = -1 \Rightarrow$ decr

$\left.\frac{dy}{dx}\right|_{y=-10} = 143 \Rightarrow$ incr

$\left.\frac{dy}{dx}\right|_{y=10} = 9 * 7 = 63 \Rightarrow$ incr



Example 31.3.5. Find and classify the equilibrium solutions of

$$\frac{dx}{dt} = x^2 - x$$

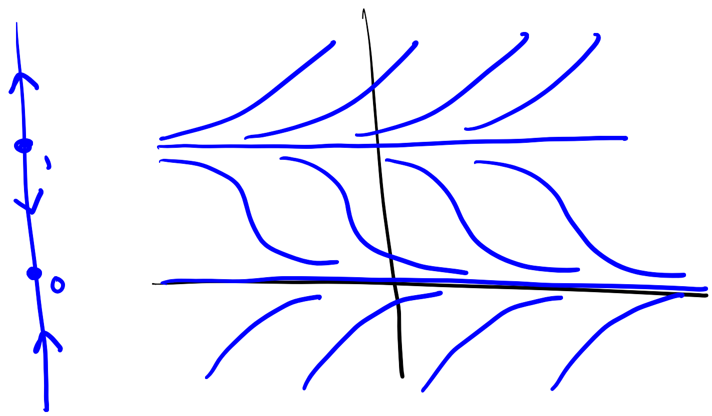
Sketch representatives of the family of solutions with x as the vertical axis and t the horizontal.

$\frac{dx}{dt} = x(x-1) = 0 \iff x = 0, 1$

if $x > 1$ $\quad \frac{dx}{dt} > 0$

if $x < 0$ $\quad \frac{dx}{dt} > 0$

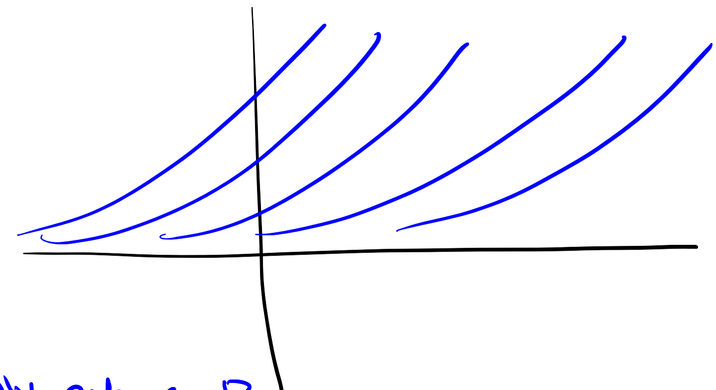
if $0 < x < 1$ $\quad \frac{dx}{dt} < 0$



Example 31.3.7. Remember, under perfect conditions, population grows at a rate proportional to itself. Suppose the number of fish in a lake grows according to the equation

$$\frac{dP}{dt} = 0.0005P.$$

Recall that the solution family to this differential equation is $P = Ce^{0.0005t}$. Graph some particular solutions to this curve. What are the equilibrium? What's the maximum population? Is this reasonable?



- only eqb is $P=0$
- no maximum population
- this seems unrealistic

Definition 31.3.8. Really, resources like space or food would limit the size of the population to some amount of fish, say L. It turns out a more reasonable model is

$$\frac{dP}{dt} = kP(L - P) = kLP - kP^2.$$

This is called **logistic population growth**, and L is called the **carrying capacity.**

Example 31.3.9. Suppose the number of fish in a lake grows according to the equation

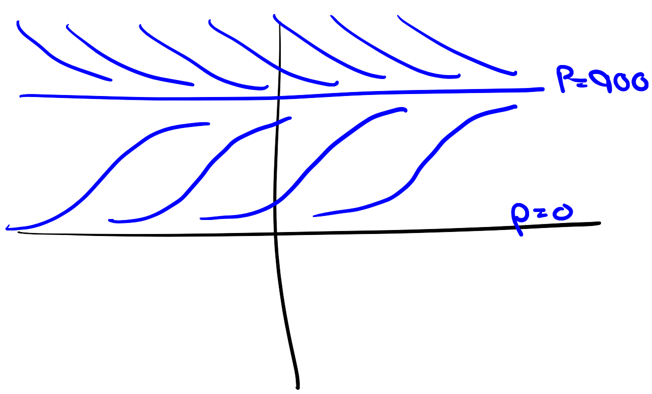
$$\frac{dP}{dt} = 0.45P - 0.0005P^2.$$

(a) What is the lake's carrying capacity for fish? Is it a stable equilibrium?

$0.45P - 0.0005P^2 = 0 \iff P = 0, 900$

carrying capacity is 900

it is a stable eqb



(b) What size is the fish population when it is growing most rapidly?

want maximum of $\frac{dP}{dt}$. Use $\frac{d^2P}{dt^2}$!

$$\frac{d^2P}{dt^2} = \frac{d}{dt}\left(0.45P - 0.0005P^2\right) = 0.45\frac{dP}{dt} - 0.001P\frac{dP}{dt}$$

$$= (0.45 - 0.001P)\frac{dP}{dt}$$ ← know this!

$$= (0.45 - 0.001P)(0.45P - 0.0005P^2)$$

$$= 0 \iff P = 0, 450, 900$$

(check this)

$\Rightarrow$ maximum at $P = 450$