MATH 118
Written Assignment #3

For the written homework assignment, you are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence of your solutions. Your solutions must have your name written on the top of the first page.

Your written assignment should be submitted as a pdf electronically through Canvas. Please email Pam if you are experiencing any technical difficulties.

2.5 Inverse Matrices

1. Recall from our "Finding the Inverse of a Matrix" workshop: a matrix obtained from the identity matrix by a single elementary row operation is called an *elementary matrix*. Determine the 3×3 matrices corresponding to
   (a) $R_1 \leftrightarrow R_2$ (this notation means swap rows 1 and 2),
   (b) $R_3 = -7R_3$
   (c) $R_2 = R_2 - 3R_1$

2. Let
$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$
   Multiplying a matrix A (on the left) by an elementary matrix has the effect of performing the same operation on A that brought I to E. For example
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+2a & h+2b & i+2c \end{bmatrix}$$
   Here, the elementary matrix on the left represents $R_3 = R_3 + 2R_1$. Write each of the matrices below as a product of an elementary matrix E (on the left) times A.
   (a) $\begin{bmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{bmatrix}$
   (b) $\begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix}$
   (c) $\begin{bmatrix} a-3d & b-3e & c-3f \\ d & e & f \\ g & h & i \end{bmatrix}$
   (d) $\begin{bmatrix} a & b & c \\ d+a & e+b & f+c \\ g & h & i \end{bmatrix}$

3. Let A be the 3×3 matrix from the previous problem. We can perform multiple row operations on A, and keep track of them using elementary matrices! Without any actual computation, can you determine the following products?

   (HINT: $E_2 E_1 A = E_2(E_1 A)$ is associative, so you might think of E_1 as the first row operation)

   (a) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =?$

   (b) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =?$

   (c) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} =?$

4. Read through our work in the "Finding the Inverse of a Matrix" workshop that we did in class on Tuesday, February 5. Use that or the questions above to explain (try your best, it's enough) why reducing A to the identity and keeping track of our operations with E_1, E_2, E_3, $\ldots, E_n$ will give us that $A^{-1} = E_n E_{n-1} \cdots E_3 E_2 E_1$.

   This explains the following algorithm for finding the inverse of a matrix:

$$\left[\begin{array}{c|c} A & I \end{array}\right] \sim \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

5. We learned that a square matrix A is invertible if and only if it is row equivalent (can be reduced) to the identity matrix. All of the following are equivalent to A being invertible. *Briefly* explain why (no full sentences necessary). This helps to see if a matrix is invertible without reducing all the way to the identity.

   (a) when reducing A, we find no zero rows
   (b) when reducing A, we find no zero columns (columns with all zeros)
   (c) there's a pivot in every row
   (d) there's a pivot in every column
   (e) $A\mathbf{x} = \mathbf{b}$ has a (unique!) solution for every $\mathbf{b}$
   (f) the augmented matrix for $A\mathbf{x} = \mathbf{b}$ has no free variables

6. We learned that a square matrix A is invertible if and only if it can be reduced to the identity matrix. The following are equivalent to A being ***not*** being invertible. *Briefly* explain why (no full sentences necessary). This helps to see if a matrix is not invertible at a glance.

   (a) a row is a constant multiple of another
   (b) a column is a constant multiple of another
   (HINT: think about what this would mean for the product $A\mathbf{x}$)

REMARK: You might want to collect items from questions 5 and 6 into an actual theorem. One might call it "The Invertible Matrix Theorem." These are nicer, quicker ways to determine if something is invertible or not.

7. Okay! Now some applications. Let

$$A = \begin{bmatrix} 2 & 5 \\ -4 & 3 \end{bmatrix}.$$

(a) Find A^{-1}.

(b) Solve the matrix equation $A\mathbf{x} = \mathbf{b}$ if

$$\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 7 \\ -2 \end{bmatrix}.$$

8. A large apartment building is to be built using modular construction techniques. The arrangement of apartments on each floor is to be chosen from one of three basic floor plans. Plan A has 2 three-bedroom units, 0 two-bedroom units, and 5 one-bedroom units on a floor. Plan B has 1 three-bedroom units, 2 two-bedroom units, and 2 one-bedroom units on a floor. Plan C has 1 three-bedroom units, 3 two-bedroom units, and 3 one-bedroom units on a floor.

   Use an inverse matrix to see if it is possible to design the building so that it has exactly 30 three-bedroom units, 60 two-bedroom units, and 70 one-bedroom units. If it is possible, how many floors will there be?

3.1 Vector Spaces (and a little bit of 3.2)

9. Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \\ 2 & 2 & 3 \end{bmatrix}$, $\mathbf{u} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -5 \\ 10 \\ 0 \\ 5 \end{bmatrix}$.

(a) Find a nonzero vector in $\text{Col}(A)$.

(b) Is $\mathbf{u}$ in $\text{Col}(A)$? Is $\mathbf{u}$ in $\text{Null}(A)$?

(c) Is $\mathbf{v}$ in $\text{Col}(A)$? Is $\mathbf{v}$ in $\text{Null}(A)$?

10. Let $\mathbf{u}$ and $\mathbf{v}$ be vectors in $\mathbb{R}^n$. Show that $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is a subspace of $\mathbb{R}^n$.

11. Let A be an $m \times n$ matrix. Show that $\text{Null}(A)$ is a subspace of $\mathbb{R}^n$. **Hint:** You may wish to use the following properties of matrix multiplication:

- $A(\mathbf{x}_1 + \mathbf{x}_2) = A\mathbf{x}_1 + A\mathbf{x}_2$ for any vectors $\mathbf{x}_1$, $\mathbf{x}_2$ in $\mathbb{R}^n$.
- $A(c\mathbf{x}) = c(A\mathbf{x})$ for any vector $\mathbf{x}$ in $\mathbb{R}^n$ and any number c.

12. Match each algebraic description of a set of vectors with its geometric picture. Determine whether or not each set is a subspace of $\mathbb{R}^n$, and if it isn't, explain why not.

(a) $\{(x, y) \mid x^2 + y^2 < 1\}$

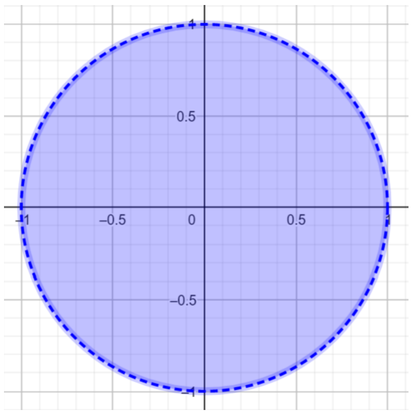
(b) $\{(x, y) \mid x > 0 \text{ and } y > 0\}$

(c) $\{(x, y, z) \mid x + y + z = 0\}$

(d) $\{(x, y, z) \mid x + y + z = 1\}$

(e) $\{(x, y, z) \mid y = x + 1 \text{ and } z = 0\}$

(f) $\{(x, y, z) \mid y = x \text{ and } z = 0\}$



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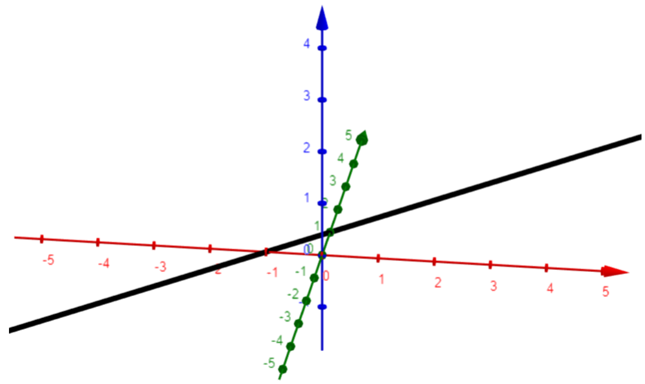

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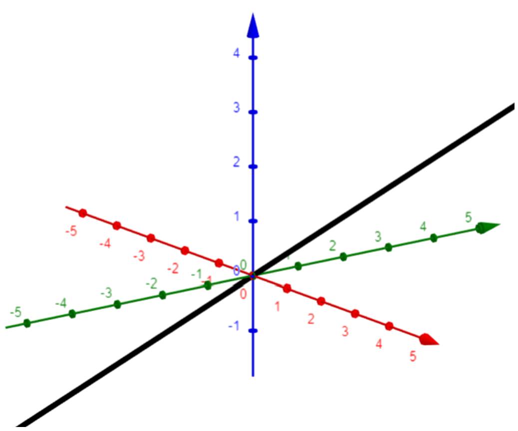

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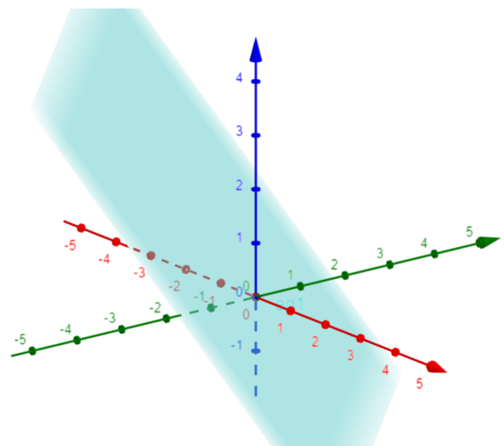

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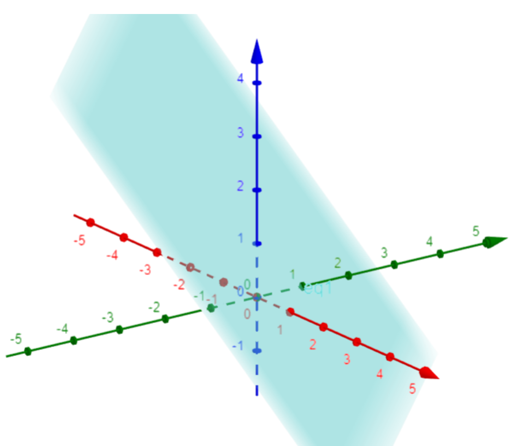

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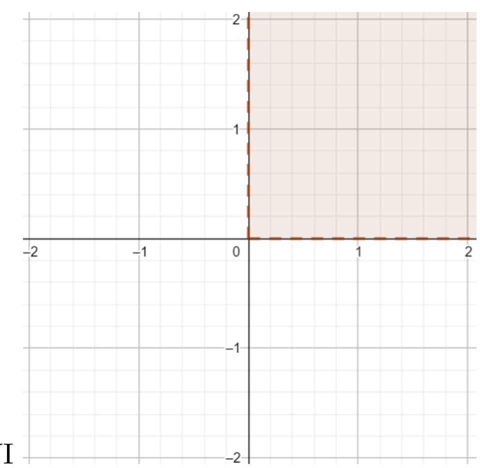

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