MATH 118

Written Assignment #4 (covers through 3.5.7 on page 62 of your notes)

For the written homework assignment, you are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence of your solutions. Your solutions must have your name written on the top of the first page.

This set looks long, but there's actually very little for you to write!

3.4. Geometry of $A\mathbf{x} = \mathbf{0}$ and $A\mathbf{x} = \mathbf{b}$

1. If A is an $m \times 3$ matrix, then the solution set to $A\mathbf{x} = \mathbf{0}$ is a line through the origin precisely when A has one free variable. Suppose we want to actually graph this line. On a computer, head over to `https://www.geogebra.org/3d`. Only part (c) requires you to write anything.

   (a) This part doesn't require you to write anything. We can graph lines in (3D!) Geogebra through the origin by parameterizing them. The line through the origin and parallel to the vector (a, b, c) can be graphed by typing `(a,b,c)t` (literally, the *span* of (a, b, c)!). Play with this by graphing lines through the origin and

   i. $(1, 2, 3)$
   ii. $(-1, 1, 0)$
   iii. $(1, 1, 1)$

   (b) This part doesn't require you to write anything. We can graph lines ***not*** through the origin by specifying a point they go through, and a vector that they are parallel to. For example, if our line is parallel to the vector (a, b, c) and goes through (p, q, r), then we can use the command `(p,q,r)+(a,b,c)t`. Play with this by graphing the line

   i. parallel to the vector $(1, 2, 3)$, through the point $(1, 1, 1)$
   ii. parallel to the vector $(-1, 1, 0)$, through the point $(1, 2, 3)$
   iii. parallel to the vector $(1, 1, 1)$, through the point $(1, 0, 1)$

   (NOTE: in both (a) and (b), we're using parametric vector form to plot in Geogebra!)

   (c) Describe and compare the solution sets of $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ if

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 7 \\ -6 \end{bmatrix}.$$

   Write your answer in parametric vector form, and use this to graph them in Geogebra.

2. If A is an $m \times 3$ matrix, then the solution set to $A\mathbf{x} = \mathbf{0}$ is a plane precisely when A has two free variables. Suppose we want to actually graph this plane. On a computer, head over to `https://www.geogebra.org/3d`. Only part (b) requires you to write anything.

   (a) This part doesn't require you to write anything. We can graph planes in Geogebra through the by typing in their equation `ax+by+cz=d`. Play with this by graphing

   i. $x + 2y + 3z = 1$

ii. $x + y + z = 0$

iii. $x + 2y - z = -1$

(b) Describe and compare the solution sets to the homogeneous equation $10x - 3y - 2z = 0$, and the related equation $10x - 3y - 2z = 5$. Write the solutions to each in parametric vector form, and graph them in Geogebra.

3.5(a) Linear Independence (through Ex. 3.5.6)

3. Remember, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$ are *linearly independent* if the only solution to

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

is the trivial one (where all x_i are zero), and *linearly dependent* otherwise. Determine if each of the following sets is linearly independent or dependent. In (a) and (b), if the set is dependent, find a dependence relationship.

(a) $u = (1, 1, -1),\ v = (2, -3, 1),\ w = (8, -7, 1)$.

(b) $u = (1, -2, -3),\ v = (2, 3, -1),\ w = (3, 2, 1)$.

(c) $u = (a_1, a_2),\ v = (b_1, b_2),\ w = (c_1, c_2)$.

4. In this question, we'll learn how to tell, in some cases, linearly dependence at a glance.

(a) Justify the following statements.

i. *"If a set contains more vectors than there are entries in each vector, then the set is linearly dependent."* In other words, if I have p vectors in $\mathbb{R}^n$, and $p > n$, then they have to be linearly dependent.
(HINT: Think about the associated homogeneous system and its augmented matrix)

ii. "$\mathbf{v}_1$, $\mathbf{v}_2$ *are linearly dependent if and only if they are scalar multiples of each other."*
(HINT: Dependence means $c\mathbf{v}_1 + d\mathbf{v}_2 = \mathbf{0}$. Solve for one of the $\mathbf{v}_i$.)

iii. *"If a set of vectors contains* $\mathbf{0}$*, then it is linearly dependent."*
(HINT: Dependence means $c_1\mathbf{0} + c_2\mathbf{v}_2 + \cdots c_n\mathbf{v}_n = \mathbf{0}$. Is there a nontrivial solution?)

(b) Be careful with applying statement (ii)! If there are more than two vectors, just knowing they aren't scalar multiples doesn't mean they are linearly independent. Explain why

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \text{ and } \begin{bmatrix} 5 \\ 7 \\ 9 \end{bmatrix}$$

are linearly dependent even though no vector is a scalar multiple of the other.

(c) Use the statements in part (a) to determine by inspection (without matrices) if given sets are linearly dependent.

i. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ ii. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}$ iii. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 7 \end{bmatrix}$

iv. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ v. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ vi. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$

NOTE: There are quite a few ways to graph planes. We'll need some of them later on.

- The span of vectors (a_1, b_1, c_1) and (a_2, b_2, c_2) is just the plane through the origin and these points. To graph the span of two vectors in Geogebra, just use the command

  ```
  Plane((0,0,0),(a1,b1,c1),(a2,b2,c2))
  ```

  Here, you're just specifying three points that have to be on the plane.

- If you're given parametric vector form for a solution set, say $\mathbf{x} = \mathbf{p} + t_1\mathbf{v}_1 + t_2\mathbf{v}_2$, then you can use the lines $\mathbf{p} + t\mathbf{v}_1$ and $\mathbf{p} + t\mathbf{v}_2$ to graph them. For example, if

  $$\mathbf{x} = \begin{bmatrix} p \\ q \\ r \end{bmatrix} + t_1 \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} + t_2 \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

  Then we can use the command

  ```
  Plane((p,q,r)+(a1,b1,c1)t,(p,q,r)+(a2,b2,c2)t)
  ```