MATH 118
Written Assignment #5 (covers chapters 4.1 - 4.3 in your notes)

For the written homework assignment, you are expected to provide full solutions with complete justifications. You will be graded on the mathematical, logical and grammatical coherence of your solutions. Your solutions must have your name written on the top of the first page.

I recommend using a calculator for the computation on this assignment. *There will be no penalty in doing so, as long as you show each step. I still want you to show me A^TA, $A^T\mathbf{b}$, $\widehat{\mathbf{x}}$, etc. The calculator I use is at* `https://matrix.reshish.com/`.

Orthogonality and Projections

> **Definition.** Let V be a vector space and W be a subspace of V. The *orthogonal complement* of a vector space W is the set of all vectors that are orthogonal to W, and is denoted $V^\perp$ (typically read as "vee perp"). In terms of the dot product
>
> $$V^\perp = \{\mathbf{v} \in V \text{ such that } \mathbf{v} \cdot \mathbf{w} = 0 \text{ for every } \mathbf{w} \text{ in } W\}.$$

1. Let $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$ and $A = \begin{bmatrix} 2 & 0 & -1 & 3 \\ 4 & 2 & 3 & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$. Is $\mathbf{u}$ in $(\text{Col}(A))^\perp$?

2. Let $W = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$ where $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$.

   (a) Find a vector $\mathbf{u}$ in $W^\perp$.

   (b) Does $\{\mathbf{u}, \mathbf{w}_1, \mathbf{w}_2\}$ form a basis for $\mathbb{R}^3$?

3. Let $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

   (a) Find $\text{proj}_{\mathbf{v}}(\mathbf{u})$ (that is, find the projection of $\mathbf{u}$ onto $\mathbf{v}$).

   (b) Use (a) to find a vector orthogonal to $\mathbf{v}$.

4. Consider the point $P = (1, 2, 3)$ and the plane $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} \right\}$.

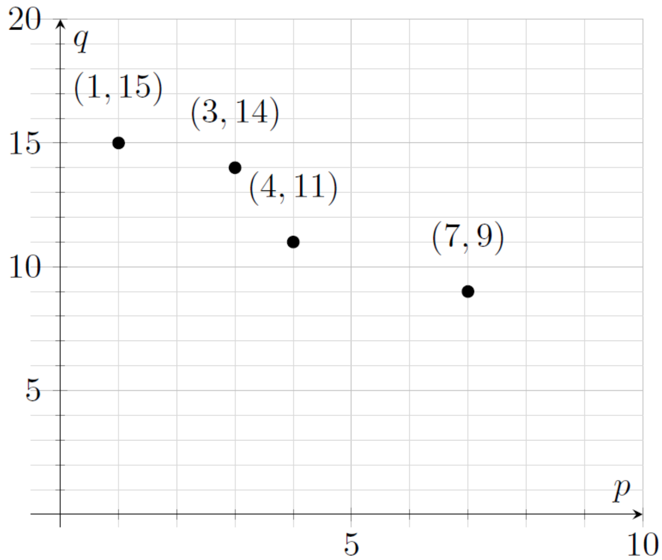
   (a) Find the closest point in W to $\mathbf{u}$.

   (b) What is the distance between $\mathbf{u}$ and W?

   (c) Check your answer using my "projection calculator" (link at end of problem set).

Least Squares

5. A coffee company would like to better understand how the demand of coffee q (in thousands of lbs) is related to the unit price of coffee p (in \$/lb). So far, all they have is some data (illustrated below).



For economic reasons, they expect p and q to satisfy a linear relationship, i.e.

$$q = cp + d;$$

for some constants c and d. They'd like to find the line that is "closest" to their data to try to better understand this relationship.

(a) Use the data to generate a system of linear equations in c and d (there will be one equation for each data point).

(b) Is the system of equations in (a) consistent?

(c) Find the least-squares solution to the system of equations from (a).

(d) Add the graph of the line corresponding to the least-squares solution to the illustration above. In what sense is this the "closest" line to the data?

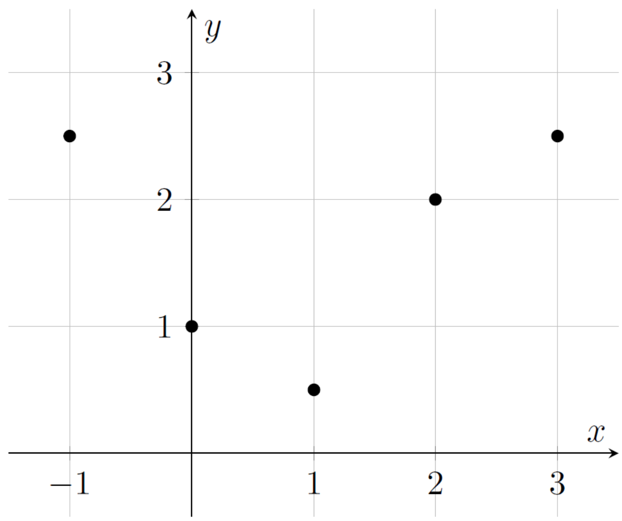
(e) Based on your model, what will be the demand if the unit price of coffee is \$10/lb? What should the company set the price of coffee to be if they want to maximize revenue?

6. Suppose you run an experiment and collect the following data.

x	y
-1	2.5
0	1
1	0.5
2	2
3	2.5

(a) You expect x and y to satisfy a linear relationship, $y = cx + d$. Use the data to generate a system of equations in c and d, and then find the least-squares solution to this system.

(b) Add the graph of the line corresponding to the least-squares solution to the data below. Do you think this linear model is a good model for the relationship between x and y?



7. After reassessing the situation in (2), you decide that it may make more sense for y to be a quadratic function of x, i.e.

$$y = a + bx + cx^2$$

for some numbers a, b, and c.

(a) Use the data to generate a system of equations in a, b, and c. Note that this system is *linear* in a, b, and c.

(b) Find the least-squares solution to the system from (a).

(c) Add the graph of your least-squares quadratic to the illustration in 2(b). Does this look like a better fit than the least-squares line?

(d) For the system in 2(a) and the system in 3(a), find each squared distance between your approximation $A\mathbf{x}$ and b, (i.e. find $\|A\mathbf{x} - \mathbf{b}\|$ in both cases). Which is smaller? Does this agree with your answer to 3(c)?

(e) Use the better of the two models to predict the value of y when $x = 6$.

8. You've been given a bunch of data relating the demand of a good q (in thousands of units) to its unit price p (in dollars per unit) shown below

p	1.0	1.5	4.5	4.8	5.3	6.5	7.7	8.0	9.5	10.5
q	29	28.5	23.5	23.0	22.1	19.7	17.1	16.9	14.5	12.8

You expect that p and q will satisfy a linear relationship, i.e.

$$q = cp + d$$

for some numbers c and d. You want to build a linear model for the relationship between p and q in order to make predictions for the demand associated with prices outside of your data set. In order to assess a model's ability to make predictions, we first *randomly* divide the data into two sub-collections, the "training set" and the "test set":

p	1.5	4.8	5.3	6.5	10.5
q	28.5	23.0	22.1	19.7	12.8

p	1.0	4.5	7.7	8.0	9.5
q	29	23.5	17.1	16.9	14.5

We then build a model using only the data from the training set, and use the unused data from the test set to assess the quality of the model's predictions.

(a) Use the data from the training set to generate a system of equations in c and d. Write this system as a matrix equation $A_1\mathbf{x} = \mathbf{b}_1$, where $\mathbf{x} = (c, d)$. Then, find the least squares solution $\widehat{\mathbf{x}}$ to this system of equations.

(b) Use the data from the test set to generate a system of equations in c and d. Write this system as a matrix equation $A_2\mathbf{x} = \mathbf{b}_2$.

(c) On average, how well does the model from (a) predict the demands associated with the prices in the test set? That is, what is the mean squared error $\frac{1}{5}\|A_2\widehat{\mathbf{x}} - \mathbf{b}_2\|$, where $\widehat{\mathbf{x}}$ is the least squares solution you found in (a)?

I made a projection calculator to check your work at `https://www.geogebra.org/m/mvzvdttm`. Here's an explanation of some of the variables you'll see
(Geogebra is picky about what I can choose for variables):

- you can change B, the point you're projecting, and A_1 and A_2, the points defining the plane
- A is the matrix $[\mathbf{a}_1\ \mathbf{a}_2]$, x_{hat} is $\widehat{x}$, p is the projection, and e is the error
- A_T is the transpose of A
- A' is $A^T A$
- b' is $A^T b$
- P is the projection matrix

In the graph, everything is labeled, but also color coded:

- the blue vectors are $\mathbf{a}_1$ and $\mathbf{a}_2$
- the vector you are projecting is black
- the projection and the error are orange