WTF analogues b/w #s and matrices
$xy = yx$ $AB \neq BA$

$xy = zy$ $\Rightarrow x = z$ $AB = CB$ $\not\Rightarrow A = C$

2.5 Inverse matrices

2.5. Key Ideas

- The inverse matrix gives $AA^{-1} = I$ and $A^{-1}A = I$.
- A is invertible if and only if it has n pivots
- If $A\mathbf{x} = 0$ for a nonzero vector $\mathbf{x}$, then A has no inverse
- The inverse of AB is $B^{-1}A^{-1}$, and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.
- Reducing $[\ A \quad I\ ]$ to reduced row echelon form gives $[\ I \quad A^{-1}\ ]$.

Definition 2.5.1. An $n \times n$ matrix A is **invertible** if there is an $n \times n$ matrix C such that

$$CA = I \text{ and } AC = I, \text{ where } I = I_n \text{ is the identity matrix.}$$

In this case, C is called the **inverse** of A. A matrix that is *not* invertible is called a **singular matrix**, and an invertible matrix is called a **non-singular matrix**.

Remark 2.5.2. Suppose B and C were both inverses of A. Then

$$B = BI = B(AC) = (BA)C = IC = C.$$

It turns out, that if A has an inverse, it's unique. We call this unique inverse A^{-1}.

Example 2.5.3. Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$ and $C = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. Show that $C = A^{-1}$

$n \times n$ ident. mtx

Theorem 2.5.9. *An $n \times n$ matrix A is invertible if and only if A is row equivalent to I_n. In this case, any sequence of elementary row operations that reduces A to I_n also transforms I_n into A^{-1}.*

Procedure 2.5.10. To find A^{-1}, row reduce the augmented matrix $[\ A \quad I\ ]$. If A is row equivalent to I, then $[\ A \quad I\ ]$ is row equivalent to $[\ I \quad A^{-1}\ ]$. Otherwise, A does not have an inverse.

if A is not row equiv to I it's NOT invertible.

Example 2.5.11. Find the inverse of $A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right]$$

check

if $A\vec{x} = \vec{b}$ and ~~A invertible~~ $\boxed{\vec{x} = A^{-1}\vec{b}}$

Usually, just reducing is faster

This is good method if want to solve

$A\vec{x} = \vec{b}$, $A\vec{x} = \vec{c}$, $A\vec{x} = \vec{d}$ ....

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 5 \\ 2 & 3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 3 \\ 0 & 5 & 0 \end{bmatrix}$$

Example 2.5.12. Poké Balls are on sale everywhere! At the new pokéstore in Pallet Town, they're selling Poké Ball bundles of 1 Poké Ball, 2 Great Balls and 2 Ultra Balls, in Veridian, the bundles consist of 2 Poké Ball, 3 Great Balls and 3 Ultra Balls, and in Pewter City, bundles come with only 5 Great Balls. By the time he gets to Mount Moon, Ash has purchased a total of 5 Poké Ball, 15 Great Balls and 10 Ultra Balls. How many bundles did he buy in each town?

(a) Find a matrix equation $A\mathbf{x} = \mathbf{b}$ whose solution, $\mathbf{x}$, represents bundles in each town.

(b) Find $\mathbf{b}$ by using the inverse of A.

pallet veridian pewter bundles in each town

$$\begin{array}{c} pb \\ gb \\ ub \end{array} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 5 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 0 \\ 2 & 3 & 5 \\ 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} -3 & 0 & 2 \\ 2 & 0 & -1 \\ 0 & 1/5 & -1/5 \end{bmatrix}$$

so

$$\vec{x} = \begin{bmatrix} -3 & 0 & 2 \\ 2 & 0 & -1 \\ 0 & 1/5 & -1/5 \end{bmatrix} \begin{bmatrix} 5 \\ 15 \\ 10 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

in pallet

in Veridian

in pewter

Example 3.1.1. Let $\mathcal{M}_{m\times n}$ be the set of all $m \times n$ matrices with entries that are real numbers. Let's focus on $\mathcal{M}_{3\times 2}$, the set of all matrices with three rows and two columns.
Let A, B, and C be matrices in $\mathcal{M}_{3\times 2}$, and c and d be real numbers.

(a) True or false: $A + B$ is another matrix in $\mathcal{M}_{3\times 2}$.

$\Rightarrow$ commutative

(b) We know $A+B = B+A$ because the entries are real numbers. For example, in the $(1,1)$-entry

$$A+B = \begin{bmatrix} a_{11} & * \\ * & * \\ * & * \end{bmatrix} + \begin{bmatrix} b_{11} & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} a_{11}+b_{11} & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} b_{11}+a_{11} & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} b_{11} & * \\ * & * \\ * & * \end{bmatrix} + \begin{bmatrix} a_{11} & * \\ * & * \\ * & * \end{bmatrix} = B+A.$$

Use the same logic to explain how we know $(A + B) + C = A + (B + C)$.

is $A+B+C = A+(B+C)$

$$A+B+C = \begin{bmatrix} a_{11}+(b_{11}+c_{11}) & * \\ * & * \\ * & * \end{bmatrix} = A+(B+C)$$

(c) How could it be possible for $A + B = A$?

yes! if $B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow$ there <u>is</u> an additive id

(d) Let Z be the 3×2 zero matrix. What can you say about the entries of A and B if $A+B = Z$?

if $A+B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$, then $a_{11} = -b_{11}$
$a_{12} = -b_{12}$
$\vdots$

i.e. everything has additive inverse

(e) By the same logic of part (b), explain how we know $c(A + B) = cA + cB$ and $c(dA) = (cd)A$.

$$c(A+B) = c\begin{bmatrix} a_{11}+b_{11} & * \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} c(a_{11}+b_{11}) & * \\ * & * \\ * & * \end{bmatrix}$$

$$= \begin{bmatrix} ca_{11}+cb_{11} & * \\ * & * \\ * & * \end{bmatrix} = cA+cB$$

(f) Is it possible for $cA = A$?

yes!
if $c=1$

What