3 Vector valued functions

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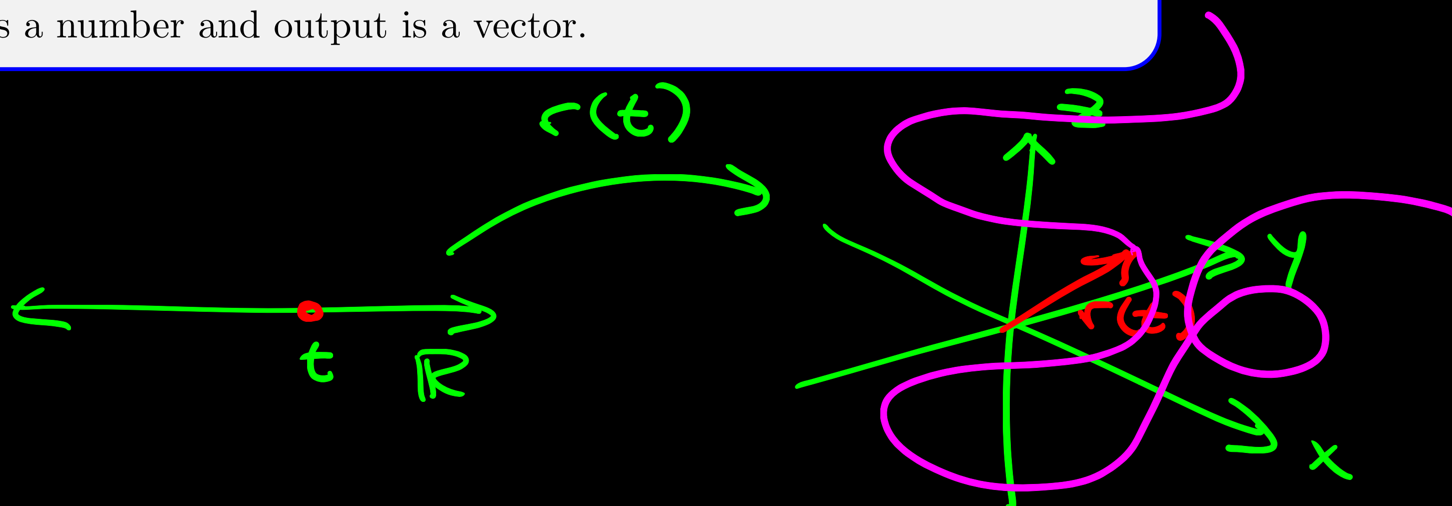
3. Key Ideas So far, the functions that we've studied in calculus have been real-valued, taking values in $\mathbb{R}$ and outputting values in $\mathbb{R}$. In this chapter, we will study functions whose outputs are vectors, primarily in three dimensions.

- define and understand vector-valued functions
- differentiate vector-valued functions
- understand what the derivative represents geometrically

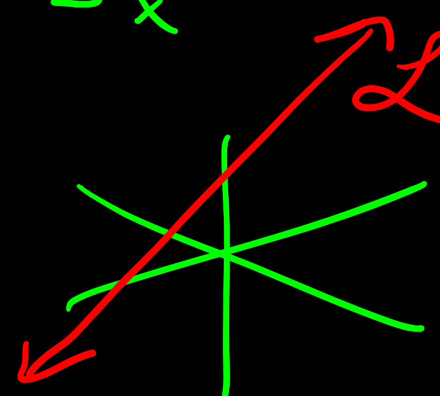
3.1 Vector valued functions

Definition 3.1. A vector-valued function is a function whose input is a number and output is a vector.



e.g. lines!

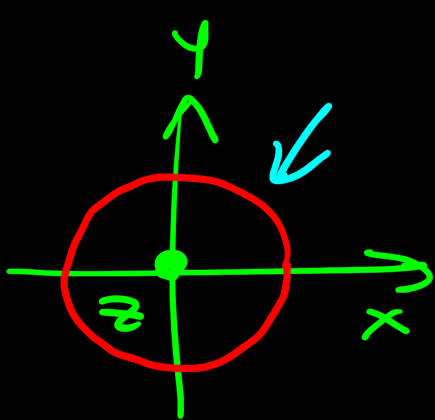
$$\mathcal{L}(t) = \langle 1,2,3 \rangle + t\langle 4,5,6 \rangle$$



Example 3.2. What do the curves
$\mathbf{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$ and $\mathbf{s}(t) = \langle \cos(t), \sin(t), t \rangle$ look like?

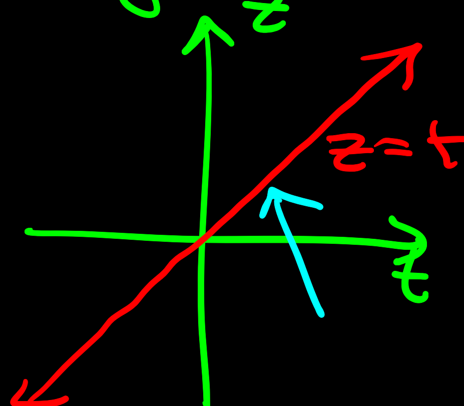
$s(t)$, we still have $x = \cos t$, $y = \sin t$

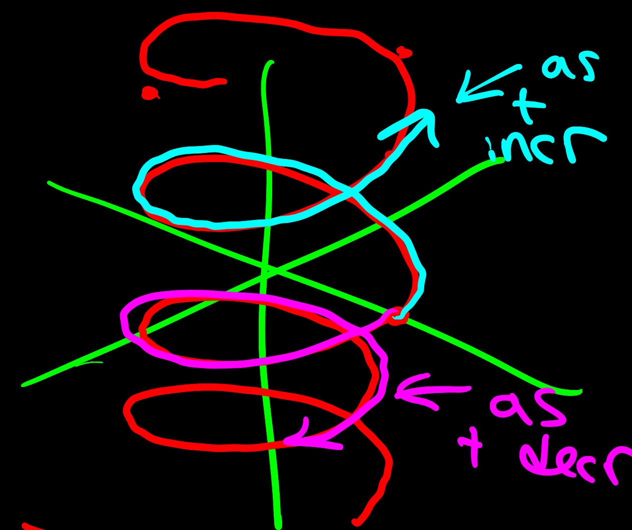
$x^2 + y^2 = 1$



top down

height





3D

$s(0) = \langle 1, 0, 0 \rangle$

$s(\pi/2) = \langle 0, 1, \pi/2 \rangle$

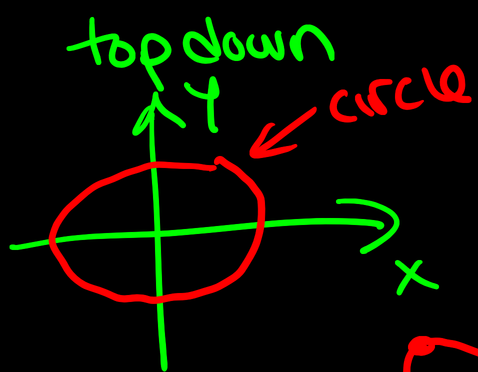
$\sin(\pi) = \langle -1, 0, \pi \rangle$

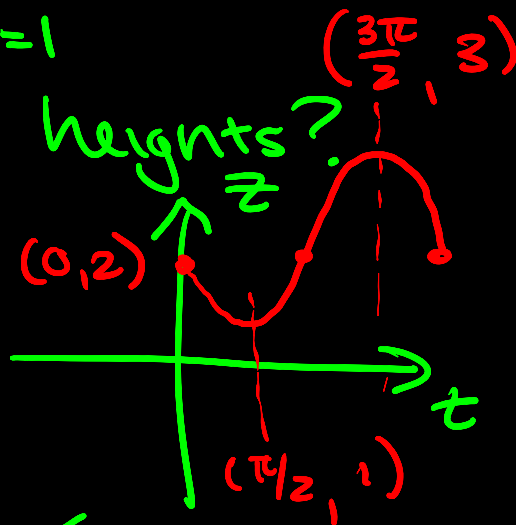
Example 3.3. What does the curve

$$x = \cos t,\ y = \sin t,\ z = 2 - \sin t \text{ for } 0 \le t \le 2\pi$$

look like?

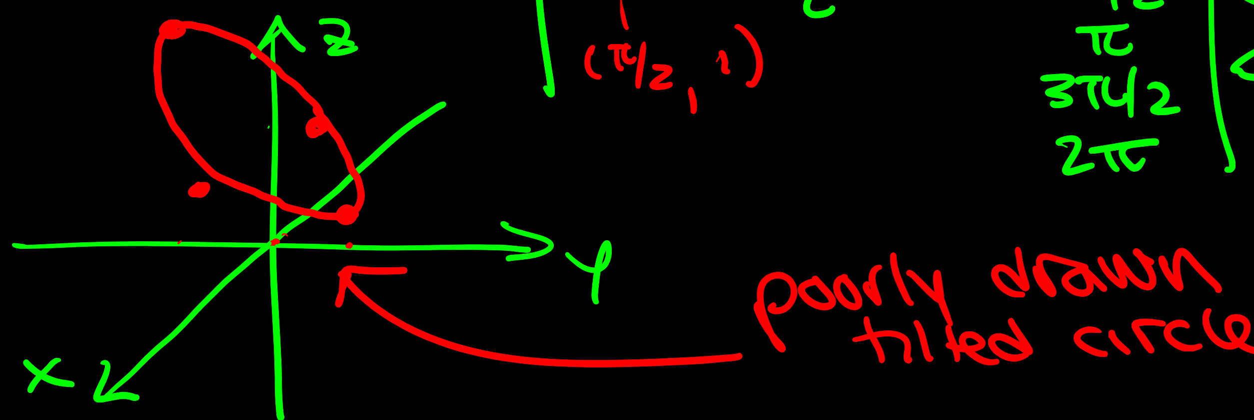
$x^2+y^2=1$



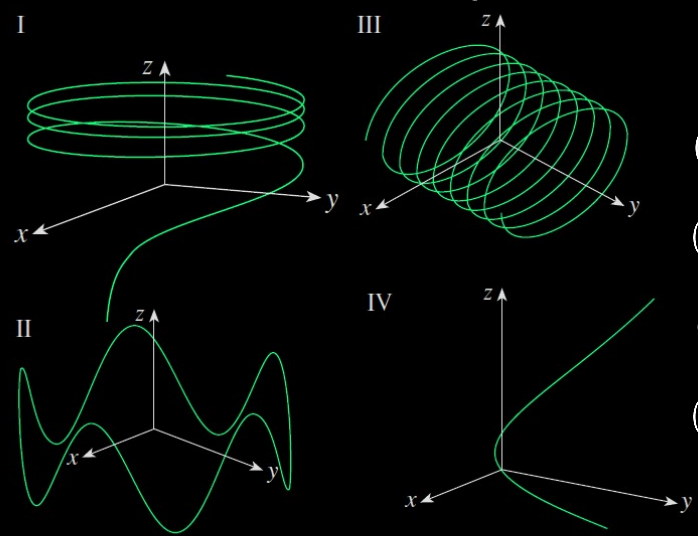


points

t	$\langle x, y, z\rangle$
0	$\langle 1, 0, 2\rangle$
$\pi/2$	$\langle 0, 1, 1\rangle$
π	$\langle -1, 0, 2\rangle$
$3\pi/2$	$\langle 0, -1, 3\rangle$
2π	$\langle 1, 0, 2\rangle$



Example 3.4. Match the graphs to their equations:



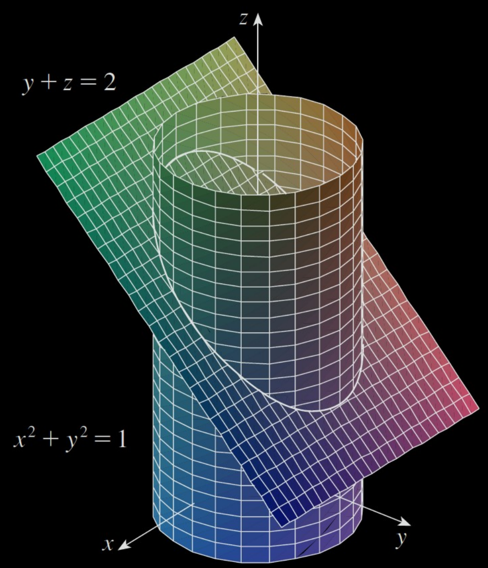
(a) $\mathbf{r}(t) = \langle \cos 4t, t, \sin 4t \rangle$

(b) $\mathbf{r}(t) = \langle \cos t, \sin t, \sin 5t \rangle$

(c) $\mathbf{r}(t) = \langle \cos t, \sin t, \ln t \rangle$

(d) $\mathbf{r}(t) = \langle t, t^2, e^{-t} \rangle$

Example 3.5. Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 2$.



lives on a cyllinder

$x^2+y^2=1$

$x=\cos t \quad \boxed{y=\sin t}$

WTK height in terms of t.

$z=2-y=2-\sin t$

$r(t)=\langle \cos t, \sin t, 2-\sin t \rangle$

3.2 Derivatives

Definition 3.6. Given a vector valued function $\mathbf{r}(t) = \langle r_1(t), r_2(t), r_3(t)\rangle$, the derivative $\mathbf{r}'(t)$ is given by

$$\mathbf{r}'(t) = \langle r_1'(t), r_2'(t), r_3'(t)\rangle$$

e.g. $r(t) = \langle t, \sin t, e^t \rangle$

$r'(t) = \langle 1, \cos t, e^t \rangle$

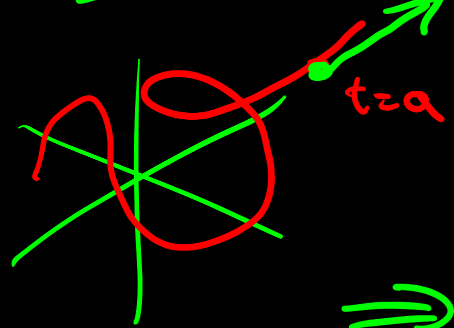
Example 3.7. If $\mathbf{r}(t) = \langle 2\cos t, \sin t, t\rangle$, find and interpret $\mathbf{r}'(t)$.

$r'(t) = \langle -2\sin t, \cos t, 1\rangle$

r(t) gives a sense of rates of change in x, y, and z dir

$r'(t) = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$

$\Rightarrow$ for small bumps in t, r'(t) gives instantaneous dir (direction of tangent to curve)



$r'(\frac{\pi}{2}) = \langle -2, 0, 1\rangle$

$\Rightarrow$ tangent at $t = \frac{\pi}{2}$ has directon vector $\langle -2, 0, 1\rangle$

Example 3.8. Find the equation of the tangent line to $\mathbf{r}(t) = \langle 2t^2, t+1, -t\rangle$ at the point $(8, 3, -2)$.

define a line we need pt. and direction

$P = (8, 3, -2)$

$r'(t) = \langle 4t, 1, -1\rangle$

$r'(2) = \langle 8, 1, -1\rangle$

direction vector

$$\boxed{\mathcal{L}(t) = \langle 8, 3, -2\rangle + t\langle 8, 1, -1\rangle}$$

$P = (8, 3, -2)$

$t = 2$ ✓

$2(2)^2 = 8$

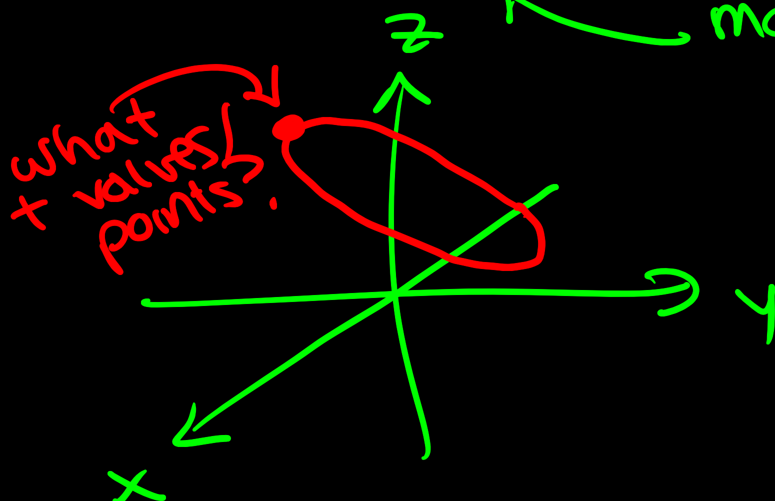
$2 + 1 = 3$

$-2 = -2$

Example 3.9. Find the point(s) at which $\langle \cos t, \sin t, 2 - \sin t \rangle$ for $0 \le t \le 2\pi$ achieves a maximum.

tilted circle above

maximum z-value



want to know when z maximized

$z = 2 - \sin t$

$\frac{dz}{dt} = 2 - \cos t$

$0 \le t \le 2\pi$

$t = \pi \Rightarrow \cos \pi = -1 \Rightarrow 2 - \cos(\pi) = 3$

So maximized at $t = \pi$ $z = 3$