Lecture 5 Derivatives

Stewart 14.1, McCallum 12.3, 12.5

Lecture 5. Key Ideas In differential calculus, we learned how to us derivatives to find tangent lines to a curve. A surface may have many tangent lines at a point, but there is only one tangent *plane*.

- understand and compute partial derivatives
- understand and compute the gradient of a function
- find tangent planes to a surface
- understand and compute second order partials
- understand and compute the Hessian of a funciton

Lecture 5.1 The partial derivative

Definition 5.1.

- The **partial derivative of** $f(x, y)$ **with respect to** x, denoted $f_x(x, y)$ or $\frac{\partial f}{\partial x}$ is the function gotten by holding y constant and differentiating with respect to x.
- The **partial derivative of** $f(x, y)$ **with respect to** y, denoted $f_y(x, y)$ or $\frac{\partial f}{\partial y}$ is the function gotten by holding x constant and differentiating with respect to y.

Example 5.2. Find the partial derivatives of $f(x, y) = x^2 \sin(y) + x$.

Example 5.3. Find the partial derivatives of $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

Definition 5.4. The **gradient** of a function $f(x, y)$, denoted $\nabla f(x, y)$ is the vector

$$\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle$$

Example 5.5. Find the gradient of $x^2 \sin(y)$ at the point $(1, 0)$.

Example 5.6. Show that

Theorem 5.7. The tangent plane to $f(x, y)$ at the point $P(x_0, y_0, z_0)$ has equation $z - z_0 = f_x(P)(x - x_0) + f_y(P)(y - y_0)$.

Example 5.8. Find the tangent plane to $f(x, y) = x^2 \sin(y)+x$ at $(x, y) = (1, 0)$

Definition 5.9. A **second order partial derivative** of a function $f(x, y)$ is one that is a partial derivative of f_x or f_y. We have

$$f_{xx} = \frac{\partial}{\partial x} f_x \quad f_{yy} = \frac{\partial}{\partial y} f_y \quad f_{xy} = \frac{\partial}{\partial y} f_x \quad f_{yx} = \frac{\partial}{\partial x} f_y$$

Example 5.10. Compute the second partials of $f(x, y) = x^2 \sin(y) + x$.

Theorem 5.11. If f_{xy} and f_{yx} are defined and continuous near a point P, then $f_{xy}(P) = f_{yx}(P)$.

Definition 5.12. The **Hessian** of a function $f(x, y)$ is

$$\text{Hess}(f) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2.$$

Example 5.13. Find the Hessians of $f(x) = x^2 + y^2$ and $g(x) = x^2 - y^2$ and evaluate them both at $(x, y) = (0, 0)$.