MATH 118

Higher order derivatives and tangent planes

1. Let $f(x,y) = x\cos(xy)$.

   (a) Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

$$\frac{\partial f}{\partial x} = 1\cdot\cos(xy) + x\cdot(-\sin(xy)\cdot(y)) = \cos(xy) - xy\sin(xy).$$

$$\frac{\partial f}{\partial y} = -x\sin(xy)\cdot(x) = -x^2\sin(xy).$$

   (b) Find all of the second partial derivatives of $f(x,y)$.

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = -y\sin(xy) - y\sin(xy) - xy^2\cos(xy)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = -x^3\cos(xy)$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = -2x\sin(xy) - x^2y\cos(xy)$$

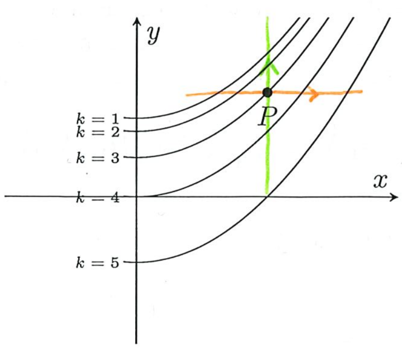
$$\frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = -x\sin(xy) - x\sin(xy) - x^2y\cos(xy)$$

EQUAL

   (c) What can you say about $\dfrac{\partial^2 f}{\partial x\partial y}$ and $\dfrac{\partial^2 f}{\partial y\partial x}$?

They are $\boxed{\text{equal.}}$

2. Use the contour plot for $f(x,y)$ below to find the sign of each of the indicated partial derivatives. at P.



   (a) $f_x > 0$ moving from P in the positive x-directions will result in an increase in height (k-value) ⇒ positive slope.

   (b) $f_y < 0$ " decrease in height ⇒ neg. slope.

   (c) $f_{xx} < 0$ contours get further apart ⇒ less steep ⇒ f_x is decreasing (since $f_x > 0$)

   (d) $f_{yy} < 0$ contours get closer together ⇒ steeper ⇒ f_y is decreasing (since $f_y < 0$)

3. If $f(x, y)$ is as in question 1, use your answer to 1 to find ∇f and Hess(f) when $x = 1$ and $y = 0$.

when $x=1, y=0$

$$\left.\frac{\partial f}{\partial x}\right|_{(1,0)} = \cos(0) - 0 = 1 \qquad \left.\frac{\partial f}{\partial y}\right|_{(1,0)} = -\sin(0) = 0 \Rightarrow \boxed{\vec{\nabla} f(1,0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

$$f_{xx}(1,0) = 0 \quad f_{yy}(1,0) = -1\cos(0) = -1 \quad f_{xy}(1,0) = 0 \quad f_{yx}(1,0) = 0 \Rightarrow \boxed{\text{Hess}(f)(1,0) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix}}.$$

4. Find the equation of the plane tangent to $f(x, y) = x^2y + e^{2x-y}$ at the point $(1, 2)$.

$$\frac{\partial f}{\partial x} = 2xy + e^{2x-y} \cdot (2) \Rightarrow f_x(1,2) = 2(1)(2) + e^{2-2} \cdot (2) = 4 + 2 = 6.$$

$$\frac{\partial f}{\partial y} = x^2 + e^{2x-y} \cdot (-1) \qquad f_y(1,2) = (1)^2 + e^{2-2} \cdot (-1) = 1 - 1 = 0.$$

$$f(1,2) = (1)^2(2) + e^{2-2} = 2 + 1 = 3$$

$\therefore$ The equation of the tangent plane is

$$z = 3 + 6 \cdot (x-1) + 0 \cdot (y-2)$$

$$\Rightarrow \boxed{z = 3 + 6(x-1)}$$

5. Find the point(s) at which the plane tangent to the surface $z = x^2 + y^2 + x^2y + 4$. is horizontal.

For the tangent plane to be horizontal, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$ at that point.

$\therefore$ ① $\frac{\partial f}{\partial x} = 2x + 2xy = 0 \Rightarrow 2x(1+y) = 0 \Rightarrow x = 0$ OR $y = -1$.

② $\frac{\partial f}{\partial y} = 2y + x^2 = 0$

If $x = 0$, ② $\Rightarrow 2y + 0^2 = 0 \Rightarrow y = 0$. ($\therefore$ one such point is $(0,0)$).

If $y = -1$, ② $\Rightarrow -2 + x^2 = 0 \rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$ ($\therefore$ $(\sqrt{2},-1)$ and $(-\sqrt{2},-1)$ are also such points).

$\therefore$ tangent plane is horizontal at $(0,0), (-\sqrt{2},-1), (\sqrt{2},-1)$.

6. The heat index (perceived temperature) I can be modelled as a function of the actual temperature T and the relative humidity H. When $T = 94°F$ and $H = 70\%$, I is measured to be $118°F$. Furthermore, $I_T(94, 70)$ is measured to be 3 and $I_H(94, 70)$ is measured to be 0.5. Estimate $I(95, 72)$.

for (T, H) close to $(94, 70)$

$$I(T,H) \approx I(94,70) + I_T(94,70)(T-94) + I_H(94,70) \cdot (H-70)$$

$$\therefore I(95,72) \approx 118 + 3(95-94) + 0.5(72-70)$$

$$= 118 + 3 + 1$$

$$= \boxed{122\,°F.}$$