Lecture 6 Directional Derivatives

Stewart 14.1, McCallum 12.3, 12.5

looked at for $f(x,y)$

partial deriv

$f_x \leftarrow$ in the x dir

$f_y \leftarrow$ in the y-dir

what about in general

Lecture 6. Key Ideas So far, we know how to compute the instantaneous rate of change of f in the direction of $\langle 1, 0\rangle$ by f_x, and $\langle 0, 1\rangle$ by f_y. What if we want to move in a different direction?

- understand and compute directional derivatives
- interpretations of directional derivatives
- properties of directional derivatives

Lecture 6.1 The gradient

Definition 6.1. The gradient of a function $f(x, y)$ is

$$\nabla f = \langle f_x, f_y \rangle.$$

The gradient of a function $f(x, y, z)$ is

$$\nabla f = \langle f_x, f_y, f_z \rangle.$$

In general, the gradient of a function $f(x_1, \ldots, x_n)$ is

$$\nabla f(x_1, \ldots, x_n) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle.$$

Definition 6.2. The **directional derivative of** $f(x, y)$ **at in the direction of** $\mathbf{u} = \langle u_1, u_2 \rangle$, denoted $D_{\mathbf{u}} f(x, y)$ is the function

$$D_{\mathbf{u}} f(x, y) = \nabla f \cdot \widehat{\mathbf{u}} = \boxed{f_x \widehat{u}_1 + f_y \widehat{u}_2.}$$

In general, the directional derivative of $f(x_1, \ldots, x_n)$ in the direction of $\mathbf{u} = \langle u_1, \ldots, u_n \rangle$ is

$$\boxed{D_{\mathbf{u}} f = \nabla f \cdot \widehat{\mathbf{u}} = f_{x_1} \widehat{u}_1 + \cdots + f_{x_n} \widehat{u}_n.}$$

also may see $D_{\hat{u}} f$

$\vec{u}$ and $\hat{u}$ are in same dir

this will rep the instantaneous ROC in the dir of $\vec{u}$.

$\hat{u}$ is normalization of $\vec{u}$

$$\hat{u} = \frac{\vec{u}}{\|\vec{u}\|}$$

$$\hat{u} = \langle \hat{u}_1, \hat{u}_2 \rangle$$

if I move in the dir of $\langle 6,-8\rangle$ IROC of z is $\frac{3+4\sqrt{3}}{5}$

Example 6.3. Find the directional derivative of $f(x, y) = e^x \cos(y)$ at $(0, \pi/3)$ in the direction of $\mathbf{u} = \langle 6, -8\rangle$.

P point start at

$\vec{u}$ direction we move in

$\frac{\partial}{\partial x} e^x \cos y$

$$D_u f(P) = \nabla f(P) \cdot \hat{u} \qquad \|u\| = \sqrt{6^2+8^2} = 10$$

$$\hat{u} = \left\langle \frac{6}{10}, \frac{-8}{10} \right\rangle = \left\langle \frac{3}{5}, \frac{-4}{5} \right\rangle$$

$$\nabla f = \langle e^x \cos y, -e^x \sin y \rangle$$

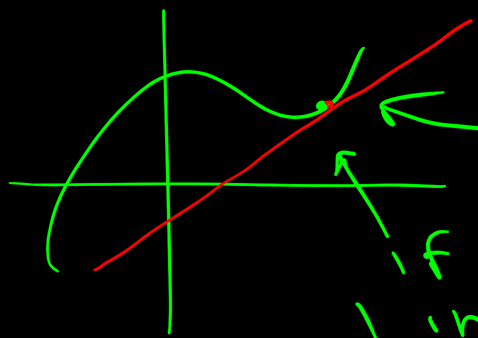
$$\nabla f(P) = \left\langle e^0 \cos \frac{\pi}{3}, -e^0 \sin \frac{\pi}{3} \right\rangle = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle$$

$$D_u f(P) = \left\langle \frac{1}{2}, -\frac{\sqrt{3}}{2} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{1}{2} * \frac{3}{5} + \frac{\sqrt{3}}{2} * \frac{4}{5}$$

$$(3+4\sqrt{3})/10$$

Example 6.4. Find and graph the equation of the tangent line to $z = x^2+y^2$ at the point $(1,1)$ in a direction parallel to $\langle 3,4\rangle$.

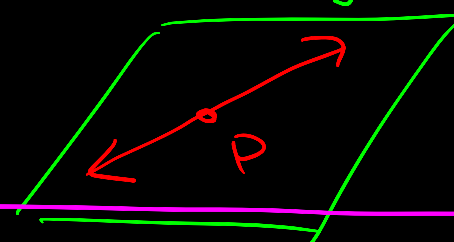
WTK IROC of z at $(1,1)$ in dir of $\langle 3,4\rangle$



$\langle 1, \frac{dy}{dx}\rangle$

if 1 more 1 in x-dir how much do I move up?

Q: why can't use $f_x(x-x_0)+f_y(y-y_0)=z-z_0$?



the the direction vector will be $\langle u_1, u_2, D_u f(p)\rangle$

$$D_u f = \langle f_x, f_y\rangle \cdot \frac{\langle 3,4\rangle}{\|\langle 3,4\rangle\|} = \langle 2x, 2y\rangle \cdot \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle = \frac{6}{5}x + \frac{8}{5}y$$

$$D_u f(1,1) = \frac{6}{5}+\frac{8}{5} = \frac{14}{5}$$

$\Longrightarrow$ direction vector of tangent is $\langle \frac{3}{5}, \frac{4}{5}, \frac{14}{5}\rangle$

$$\boxed{L_u = \langle 1,1,2\rangle + t\langle \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{14}{5}\rangle}$$

Example 6.5. Suppose the sun is centered at $(0,0,0)$. An alien space ship located at $(2,2,1)$ (measured in "docbobs") feels the following force of gravity pulling it toward the sun

$$F(x,y,z) = \frac{81}{\sqrt{x^2+y^2+z^2}}.$$

A space station is located at $(0,-2,5)$ docbobs. How will the force of gravity change if the ship begins moving straight toward the station?

P

Q

$\vec{u} = \overrightarrow{PQ} = \langle -2, -4, 4 \rangle$

$\hat{u} = \langle -\frac{1}{3}, -\frac{2}{3}, \frac{2}{3} \rangle$

If start at $(2,2,1)$ and move toward $(0,-2,5)$ how does $F(x,y,z)$ change?

$D_u F(2,2,1)$ will tell us IROC in u-dir from $(2,2,1)$

$$\nabla F = \left\langle -\frac{x}{\sqrt{x^2+y^2+z^2}}, -\frac{y}{\sqrt{x^2+y^2+z^2}}, -\frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle$$

$$\nabla F(2,2,1) = \left\langle -\frac{2}{\sqrt{2^2+2^2+1}}, -\frac{2}{3}, -\frac{1}{3} \right\rangle$$

You will investigate the following in Problem 5 of the worksheet.

Properties 6.6.

- The maximum value of $D_{\mathbf{u}}f(P)$ (that is the largest rate of change of f moving from a point P) occurs in the direction of $\nabla f(P)$, and its value is $\|\nabla f(P)\|$.
- The minimum value of $D_{\mathbf{u}}f(P)$ (that is the smallest rate of change of f moving from a point P) occurs in the direction of $-\nabla f(P)$, and its value is $-\|\nabla f(P)\|$.
- The vector $\nabla f(P)$ is perpendicular to the level curve $f(x, y) = k$ that goes through the point P.