Lecture 1 Derivatives

Stewart 14.1, McCallum 12.3, 12.5

Lecture 1. Key Ideas So far, we know how to compute the instantaneous rate of change of f in the direction of $\langle 1, 0\rangle$ by f_x, and $\langle 0, 1\rangle$ by f_y. What if we want to move in a different direction?

- understand and compute directional derivatives
- interpretations of directional derivatives
- properties of directional derivatives

Lecture 1.1 The gradient

Definition 1.1. The gradient of a function $f(x, y)$ is

$$\nabla f = \langle f_x, f_y\rangle.$$

The gradient of a function $f(x, y, z)$ is

$$\nabla f = \langle f_x, f_y, f_z\rangle.$$

In general, the gradient of a function $f(x_1, \ldots, x_n)$ is

$$\nabla f(x_1, \ldots, x_n) = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_1}\rangle.$$

Definition 1.2. The **directional derivative of** $f(x, y)$ **at in the direction of** $\mathbf{u} = \langle u_1, u_2\rangle$, denoted $D_{\mathbf{u}}f(x, y)$ is the function

$$D_{\mathbf{u}}f(x, y) = \nabla f \cdot \widehat{\mathbf{u}} = f_x\widehat{u}_1 + f_y\widehat{u}_2.$$

In general, the directional derivative of $f(x_1, \ldots, x_n)$ in the direction of $\mathbf{u} = \langle u_1, \ldots, u_n\rangle$ is

$$D_{\mathbf{u}}f = \nabla f \cdot \widehat{\mathbf{u}} = f_{x_1}\widehat{u}_1 + \cdots + f_{x_n}\widehat{u}_n.$$

Example 1.3. Find the directional derivative of $f(x, y) = e^x \cos(y)$ at $(0, \pi/3)$ in the direction of $\mathbf{u} = \langle 6, -8\rangle$.

Example 1.4. Find and graph the equation of the tangent line to $z = x^2 + y^2$ at the point $(1, 1)$ in a direction parallel to $\langle 3, 4\rangle$.

Example 1.5. Suppose the sun is centered at $(0, 0, 0)$. An alien space ship located at $(2, 2, 1)$ (measured in "docbobs") feels the following force of gravity pulling it toward the sun

$$F(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}.$$

A space station is located at $(0, -2, 5)$ docbobs. How will the force of gravity change if the ship begins moving straight toward the station?

You will investigate the following in Problem 5 of the worksheet.

Properties 1.6.

- The maximum value of $D_{\mathbf{u}}f(P)$ (that is the largest rate of change of f moving from a point P) occurs in the direction of $\nabla f(P)$, and its value is $\|\nabla f(P)\|$.
- The minimum value of $D_{\mathbf{u}}f(P)$ (that is the smallest rate of change of f moving from a point P) occurs in the direction of $-\nabla f(P)$, and its value is $-\|\nabla f(P)\|$.
- The vector $\nabla f(P)$ is perpendicular to the level curve $f(x, y) = k$ that goes through the point P.