MATH 118
Directional derivatives and the gradient

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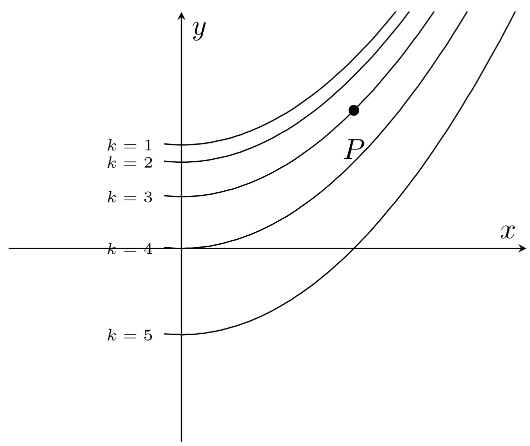
1. Let $f(x,y) = x\ln(1+y^2)$. Find $D_{\hat{\mathbf{u}}}f(2,1)$ if $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

2. Consider the contour plot for $f(x,y)$ shown below.



Use the contour plot to determine the sign of $D_{\hat{\mathbf{u}}}(P)$ for each of the following vectors $\mathbf{u}$. (Assume the x and y axes have the same scale.)

(a) $\mathbf{u} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(b) $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

3. Let $f(x, y, z) = 3x^2 + 2xz + y^2z^2$.

   (a) Find ∇f.

   (b) Find $D_{\hat{\mathbf{u}}} f(1, 2, -1)$ if $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$.

4. If $z = f(x, y)$, $D_{\hat{\mathbf{u}}} f(1, 2) = 5\sqrt{2}$ and $D_{\hat{\mathbf{v}}} f(1, 2) = -3$, where $\hat{\mathbf{u}} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{bmatrix}$ and $\hat{\mathbf{v}} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}$, what is $\nabla f(1, 2)$?

5. Through directional derivatives, we can calculate rates of change in any direction. In this exercise, we will determine the **directions in which the maximum and minimum rates of change occur**.

   (a) Let $f(x, y)$ be a differentiable function. Use the geometric property of dot products to write the directional derivative $D_{\hat{\mathbf{u}}} f$ in terms of the length of ∇f and the angle between ∇f and the unit vector $\hat{\mathbf{u}}$.

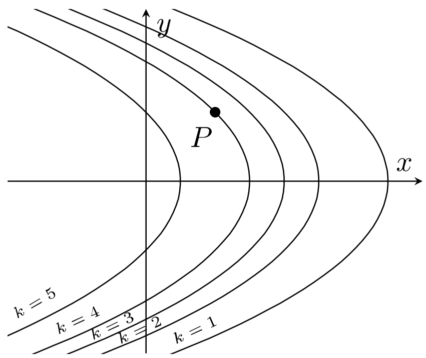
   (b) Based on your answer to (a), what is the maximum value of $D_{\hat{\mathbf{u}}} f$ at a specified point? What is the minimum value?

   (c) For which angle between ∇f and $\mathbf{u}$ does the maximum rate of change occur? The minimum?

   (d) In which direction does the maximum rate of change occur (*i.e.* which $\hat{\mathbf{u}}$ maximizes $D_{\hat{\mathbf{u}}} f$? The minimum?

6. Find the direction of the maximum rate of change ("steepest ascent") of $f(x, y) = 10 - x^2 - 2y^2$ at (1,0). Make a rough sketch of the surface and sketch in a vector in this direction. Do you think your answer makes sense?

7. Consider the contour plot of $f(x, y)$ below.

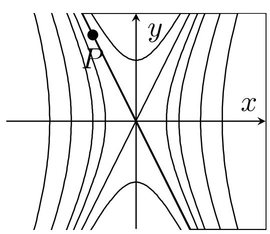


(a) Add in a vector at P (to the contour plot) that points in the direction of the maximum rate of change.

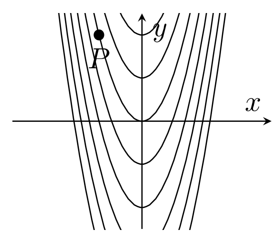
(b) Make a rough sketch of the "fastest path" through the level curves of $f(x, y)$ that passes through P.

(c) Based on your conclusion from question 5 and your vector from (a), what can you say about the direction of $\nabla f(P)$ in relation to the level curve of $f(x, y)$?

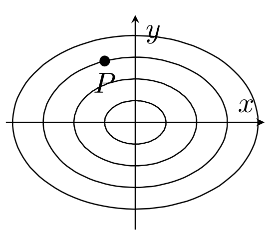
8. Given that $\nabla f(P) = \begin{bmatrix} 2 \\ -8 \end{bmatrix}$, which of the following is a contour plot for $f(x, y)$? (Assume the x and y axes have the same scale for each contour plot.)



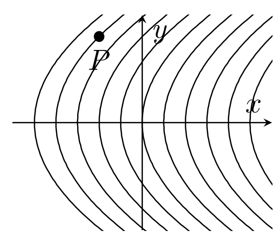
(I)



(III)



(II)



(IV)