Lecture 7 The Multivariable Chain Rule

Stewart 14.1, McCallum 12.3, 12.5

- chain rule part 1
- chain rule part 2
- chain rule general case
- implicit differentiation

Question 7.1. What do we remember about the chain rule from single variable calculus?

Example 7.2. If $z = xy$, where $x = t^2$ and $y = \sin t$, find $\frac{dz}{dt}\big|_{t=\pi}$.

(this notation means $\frac{dz}{dt}$ when $t = \pi$).

Theorem 7.3. (Chain Rule Case 1) If $z = f(x, y)$ where $x = g(t)$ and $y = h(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

Example 7.4. Find the tangent line at $t = \pi$ of $f(x, y) = xy$, where

$$x = t^2 \qquad y = \sin t.$$

Theorem 7.5. (Chain Rule Case 2) If $z = r(x, y)$ where $x = f(s, t)$ and $y = g(s, t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Example 7.6. If $z = e^x \sin y$, where $x = st^2$ and $y = s^2 t$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$.

Example 7.7. Sometimes we might have z as a function of several variables which are themselves functions of several variables, which are in turn...
For example, suppose $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, and finally $s = \phi(u, v, w)$ and $t = \psi(u, v, w)$. Draw a tree diagram, and find $\frac{\partial z}{\partial z}$.

Example 7.8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 1$, and interpret $\partial z/\partial x$ at $(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ geometrically.

Question 7.9. What do we remember about implicit differentiation from single variable calculus? Use $x^2 + y^2 = 1$ as an example.

Theorem 7.10. Suppose instead of a function $z = f(x, y)$, we are given z implicitly by an equation $F(x, y, z) = 0$. Then by the chain rule

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{\partial F/\partial x}{\partial F/\partial z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\partial F/\partial y}{\partial F/\partial z}.$$

Example 7.11. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $x^2 + y^2 + z^2 = 1$, and interpret $\partial z/\partial x$ at $(\frac{\sqrt{3}}{2}, 0, \frac{1}{2})$ geometrically.