MATH 118
The Multivariable Chain Rule

Suppose you're hiking up a mountain modelled by the surface $z = f(x, y)$. You walk on a hiking trail such that your x and y-coordinates are functions of time. This means that your elevation can also be expressed as a function of time. To find the rate of change of your elevation as a function of time (i.e., the slope of the hiking trail), we need the *Multivariable Chain Rule*.

1. (a) If $z = f(x, y)$, $x = g(t)$ and $y = h(t)$, draw the tree diagram for z.

   (b) Write down the formula for $\frac{dz}{dt}$.

   (c) If $f(x, y) = x^2 + 3xy^2 + 2y$, $g(t) = \cos(t)$ and $h(t) = \sin(3t)$, use part (a) to find $\frac{dz}{dt}$ when $t = 0$.

2. (a) In certain situations, x and y may themselves be functions of two new variables. If $z = f(x, y)$, $x = g(s, t)$ and $y = h(s, t)$, draw the tree diagram for z.

(b) Write down the formula for $\dfrac{\partial z}{\partial s}$.

(c) If $f(x, y) = x^2 + 3xy^2 + 2y$ (as before) and now $g(s, t) = s\cos(t)$, $h(s, t) = s^2 \sin(3t)$, use part (a) to find $\dfrac{\partial z}{\partial s}$ when $(s, t) = (1, \pi)$.

3. (a) In other situations, the new variables s and t may themselves depend on other variables u and v. If $z = f(x, y)$, $x = g(s, t)$, $y = h(s, t)$, $s = \phi(u, v)$ and $t = \psi(u, v)$, draw the tree diagram for z.

(b) Write down a formula for $\dfrac{\partial z}{\partial u}$.

4. The temperature at a point (x, y) is $T(x, y)$, measured in degrees Celsius. A bug crawls so that its position after t seconds is given by $x = \sqrt{1+t}$, $y = 2 + \frac{1}{3}t$, where x and y are measured in centimetres.

(a) For each of the following, provide the units of the rate of change and write a sentence interpreting its meaning.

(i) $\dfrac{\partial T}{\partial x}$

(ii) $\dfrac{\partial T}{\partial y}$

(iii) $\dfrac{dx}{dt}$

(iv) $\dfrac{dy}{dt}$

(v) $\dfrac{dT}{dt}$

(b) If the temperature function satisfies $T_x(2, 3) = 4$ and $T_y(2, 3) = 3$, how fast is the temperature rising on the bug's path after 3 seconds?

5. Suppose $f(x, y)$ is a differentiable function of x and y, and $g(u, v) = f(e^u + \sin(v), e^u + \cos(v))$. Use the table of values to calculate $g_u(0, 0)$ and $g_v(0, 0)$.

(x, y)	f	g	f_x	f_y
$(0, 0)$	3	6	4	8
$(1, 2)$	6	3	2	5

6. Consider the ellipsoid $x^2 + 2y^2 + 3z^2 = 4$.

(a) Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ at the point $(1, 0, 1)$.

(b) Use your answer to (a) to find the equation of the plane tangent to the ellipsoid at $(1, 0, 1)$.