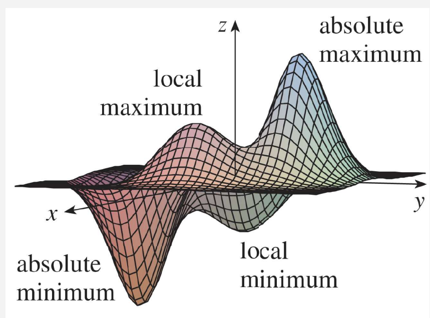
Lecture 8 Local extrema

Stewart 14.1, McCallum 12.3, 12.5

- second derivative test
- extrema from contour maps

Question 8.1. In single variable calculus: how did we look for local (relative) extrema?

Definition 8.2. $f(x, y)$ has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ for all points near (a, b). $f(x, y)$ has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ for all points near (a, b).

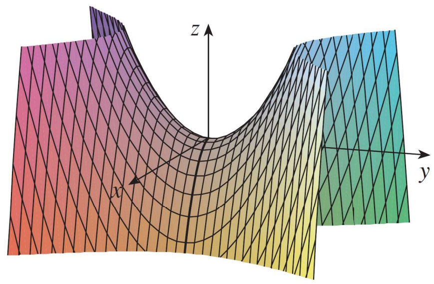


Theorem 8.3. If f has a local extremum at (a, b) and f_x and f_y exist there, then $f_x(a, b) = f_y(a, b) = 0$.

Definition 8.4. The point $P(a, b)$ is called a **critical point of** f if $f_x(a, b) = f_y(a, b) = 0$.

Example 8.5. Find the critical point(s) of $f(x, y) = x^2 + y^2 - 2x - 6y + 14$.

Example 8.6. Find the critical point of $f(x, y) = y^2 - x^2$. Is this a relative extremum?

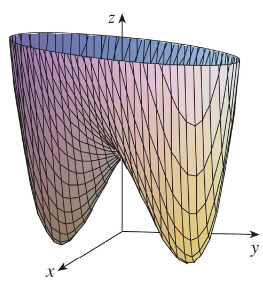


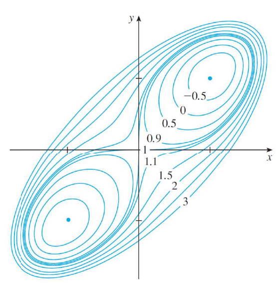
Theorem 8.7. If (a, b) is a critical point of $f(x, y)$ (i.e. if $\nabla f(a, b) = 0$ or is undefined, then let

$$D(a, b) = \begin{vmatrix} f_{xx}(a, b) & f_{xy}(a, b) \\ f_{xy}(a, b) & f_{yy}(a, b) \end{vmatrix} = f_{xx}(a, b) f_{yy}(a, b) - \big(f_{xy}(a, b)\big)^2$$

(a) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum
(b) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local minimum
(c) If $D < 0$, then $f(a, b)$ is a saddle point

Example 8.8. Find the local extrema and saddle points of $f(x, y) = x^4 + y^4 - 4xy + 1$.

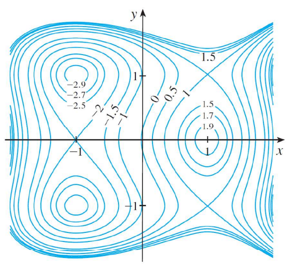




Poll 8.9. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

How many critical points does f have?

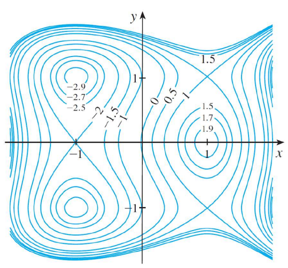


(a) 0
(b) 3
(c) 6
(d) I don't know.

Poll 8.10. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

Classify the extrema of f.

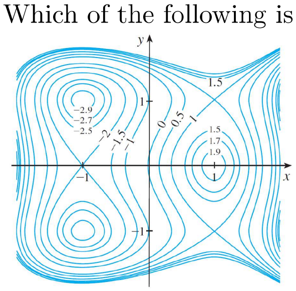


(a) f has three local maximums and no local minimums
(b) f has two local maximums and one local minimum
(c) f has three local minimums and no local maximum
(d) f has two local minimums and one local maximum

Poll 8.11. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

Which of the following is ***not*** a saddle point of f?

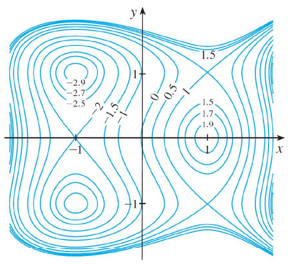


(a) $(-1, 0)$
(b) $(1, 1)$
(c) $(-1, 1)$
(d) $(1, -1)$

Poll 8.12. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

Which is a reasonable guess for the absolute minimum of f in this window?



(a) $f(x, y) = -3$
(b) $f(x, y) = 1$
(c) $f(x, y) = -3.4$
(d) $f(x, y) = -1.6$

Example 8.13. The contour map below is for the surface

$$f(x, y) = 3x - x^3 - 2y^2 + y^4.$$

Use the map to classify the relative extrema and find saddle points of f, then confirm your prediction analytically.